Exam for „Algorithms I“

SS 2022 / August 18, 2022

First name: ____________________

Second name: ____________________

Matriculation number: ____________________

task	max. points	points achieved
1	4	
2	6	
3	8	
4	6	
5	8	
6	8	
Σ	40	

Important information

- Duration of the exam: **60 minutes**.
- Tools: You are allowed to use a sheet of paper (size DIN A4). Both sides can be written on by hand (no printed paper).
- Write with an indelible pen. Do not write in red paint.
- Check the exam you have been given for completeness: 6 tasks on 6 pages.
- Enter your name and matriculation number in the appropriate fields on each sheet.
- Write your solutions in the spaces provided. If there is not enough space in a field, use the back of the corresponding sheet and indicate this on the front. If there is still not enough space, you can ask the supervisor for additional sheets of paper.

Task 1. (4 Points)

Which of the following statements hold for all functions $f : \mathbb{N} \to \mathbb{N}$ (and $g : \mathbb{N} \to \mathbb{N}$)? If a statement is not correct, provide a counterexample in form of concrete functions f and g.

1. If $f \in \mathcal{O}(g)$ then $f \in o(g)$.
2. $f \in \mathcal{O}(g)$ or $g \in \mathcal{O}(f)$.
3. $f \in \mathcal{O}(f^2)$
4. $f \in o(f^2)$

No proof is required for correct statements.

Solution:

1. not correct: take for instance $f(n) = n$ and $g(n) = 2n$
2. not correct: take for instance $f(n) = n$ and $g(2n) = \lceil \log(n) \rceil$ for even and $g(2n+1) = n^2$ for odd n.
3. correct since $f(n) \leq f(n)^2$
4. not correct: take for instance $f(n) = 1$ for all n.

Task 2. (6 Points)

Consider the following pseudo code for a recursive function f.

function $f(n : \text{integer})$
  let m be the smallest number of the form 3^k with $3^k \geq n$;
  if $m = 1$ **then print**(goodbye)
  else
    for $i = 1$ to m^2 **do**
      print(hello)
    endfor
    $m := m/3$;
    for $i = 1$ to 8 **do**
      $f(m)$
    endfor
  endif
endfunction

1. Give a recursive equation for the running time of the algorithm for f.
2. Use the Master Theorem I to compute the running time of the algorithm.

Solution:

1. $T(n) = 8 \cdot T(n/3) + \mathcal{O}(n^2)$
2. We have $a = 8$, $b = 3$ and $c = 2$. Since $a = 8 < 3^2 = b^c$, the Master theorem I yields $T(n) \in \Theta(n^2)$.

Task 3. (8 Points)

- Write down the the pseudo code for the procedure **build-heap** from heapsort.
- Consider the following array of numbers:

$$[8, 2, 4, 18, 3, 20, 21, 1]$$

  Use the procedure **build-heap** from the lecture in order to transform the array into a max-heap. Write down the sequence of swap(i, j) operations and the new array after each swap (note that i and j are array indices, i.e., numbers from $\{1, \ldots, 8\}$).
- Draw the tree structure that corresponds to the heap that you have computed in the previous point.

Solution:

- See slides
- Step 1: swap$(3, 7)$. The new array is

$$[8, 2, 21, 18, 3, 20, 4, 1]$$

  Step 2: swap$(2, 4)$. The new array is

$$[8, 18, 21, 2, 3, 20, 4, 1]$$

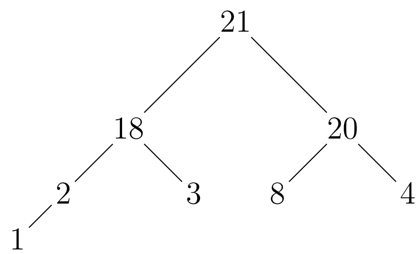
  Step 3: swap$(1, 3)$. The new array is

$$[21, 18, 8, 2, 3, 20, 4, 1]$$

  Step 4: swap$(3, 6)$. The new array is

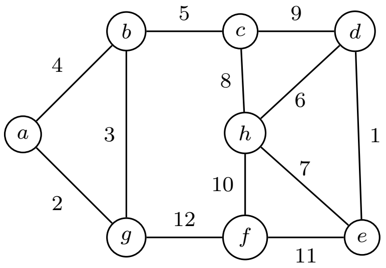
$$[21, 18, 20, 2, 3, 8, 4, 1]$$

- The max-heap is:



Task 4. (6 Points)

Compute a spanning subtree of minimal weight using Kruskal's algorithm for the following graph.



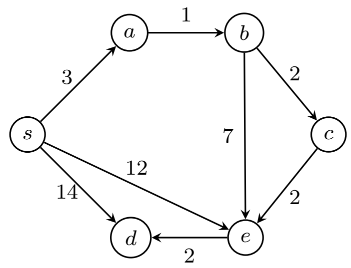
Show the edge selected in each step or indicate if no edge is selected in a step.

Solution: Let T be the current set of selected edges. Initially we have $T = \emptyset$.

1. Select edge $\{d, e\}$.
$$T = \{\{d, e\}\}.$$
2. Select edge $\{a, g\}$.
$$T = \{\{d, e\}, \{a, g\}\}.$$
3. Select edge $\{b, g\}$.
$$T = \{\{d, e\}, \{a, g\}, \{b, g\}\}.$$
4. Edge $\{a, b\}$ is not selected.
5. Select edge $\{b, c\}$.
$$T = \{\{d, e\}, \{a, g\}, \{b, g\}, \{b, c\}\}.$$
6. Select edge $\{d, h\}$.
$$T = \{\{d, e\}, \{a, g\}, \{b, g\}, \{b, c\}, \{d, h\}\}.$$
7. Edge $\{e, h\}$ is not selected.
8. Select edge $\{c, h\}$.
$$T = \{\{d, e\}, \{a, g\}, \{b, g\}, \{b, c\}, \{d, h\}, \{c, h\}\}.$$
9. Edge $\{c, d\}$ is not selected.
10. Select edge $\{f, h\}$.
$$T = \{\{d, e\}, \{a, g\}, \{b, g\}, \{b, c\}, \{d, h\}, \{c, h\}, \{f, h\}\}.$$
11. Edge $\{e, f\}$ is not selected.
12. Edge $\{f, g\}$ is not selected.

Task 5. (8 Points)

Use Dijkstra's algorithm to compute all shortest paths starting at node s in the following graph.



Show the values of the program variables B, R, U, p, D after each iteration of the main **while**-loop of Dijkstra's algorithm.

Solution:

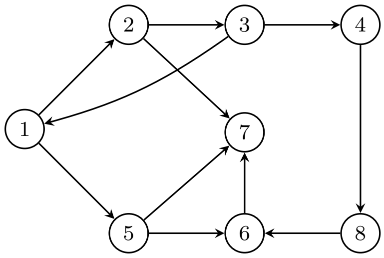
1. After iteration 1:
   $B = \{s\}$, $R = \{a, d, e\}$, $U = \{b, c\}$
   $p[a] = p[d] = p[e] = s$, $p[b] = p[c] = \text{nil}$
   $D[s] = 0$, $D[a] = 3$, $D[d] = 14$, $D[e] = 12$, $D[b] = D[c] = \infty$
2. After iteration 2:
   $B = \{s, a\}$, $R = \{b, d, e\}$, $U = \{c\}$
   $p[a] = p[d] = p[e] = s$, $p[b] = a$, $p[c] = \text{nil}$
   $D[s] = 0$, $D[a] = 3$, $D[b] = 4$, $D[d] = 14$, $D[e] = 12$, $D[c] = \infty$
3. After iteration 3:
   $B = \{s, a, b\}$, $R = \{c, d, e\}$, $U = \emptyset$
   $p[a] = p[d] = s$, $p[b] = a$, $p[c] = b$, $p[e] = b$
   $D[s] = 0$, $D[a] = 3$, $D[b] = 4$, $D[c] = 6$, $D[d] = 14$, $D[e] = 11$
4. After iteration 4:
   $B = \{s, a, b, c\}$, $R = \{d, e\}$, $U = \emptyset$
   $p[a] = p[d] = s$, $p[b] = a$, $p[c] = b$, $p[e] = c$
   $D[s] = 0$, $D[a] = 3$, $D[b] = 4$, $D[c] = 6$, $D[d] = 14$, $D[e] = 8$
5. After iteration 5:
   $B = \{s, a, b, c, e\}$, $R = \{d\}$, $U = \emptyset$
   $p[a] = s$, $p[b] = a$, $p[c] = b$, $p[d] = e$, $p[e] = c$
   $D[s] = 0$, $D[a] = 3$, $D[b] = 4$, $D[c] = 6$, $D[d] = 10$, $D[e] = 8$
6. After iteration 6:
   $B = \{s, a, b, c, d, e\}$, $R = \emptyset$, $U = \emptyset$
   $p[a] = s$, $p[b] = a$, $p[c] = b$, $p[d] = e$, $p[e] = c$
   $D[s] = 0$, $D[a] = 3$, $D[b] = 4$, $D[c] = 6$, $D[d] = 10$, $D[e] = 8$

It suffices to use the table representation (if it is clear what B, R, U, p and D are).

Node	s	a	b	c	d	e
Step 0	**0**	3	∞	∞	14	12
$p(x)$	nil	s	nil	nil	s	s
Step 1	**0**	**3**	4	∞	14	12
$p(x)$	nil	s	a	nil	s	s
Step 2	**0**	**3**	**4**	6	14	11
$p(x)$	nil	s	a	b	s	b
Step 3	**0**	**3**	**4**	**6**	14	8
$p(x)$	nil	s	a	b	s	c
Step 4	**0**	**3**	**4**	**6**	10	**8**
$p(x)$	nil	s	a	b	e	c
Step 5	**0**	**3**	**4**	**6**	**10**	**8**
$p(x)$	nil	s	a	b	e	c

Task 6. (8 Points)

- Write down Warshall's algorithm (pseudo code suffices).
- Write down the adjacency matrix A for the following directed graph:



- Compute the transitive closure of A. It suffices to give the final result.

Solution:

- See slides
- We have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- We have

$$A^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$