Submonoids and rational subsets of groups with infinitely many ends

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Abstract. In this paper we show that the membership problems for finitely generated submonoids and for rational subsets are recursively equivalent for groups with two or more ends.

1 Introduction

Let G be a finitely generated group with finite generating set Σ. Put $\widetilde{\Sigma} = \Sigma \cup \Sigma^{-1}$ and let $\pi \colon \widetilde{\Sigma}^* \to G$ be the canonical projection from the free monoid on $\widetilde{\Sigma}$ to G. The (uniform) submonoid membership problem for G takes as input words $w, w_1, \ldots, w_n \in \widetilde{\Sigma}^*$ and asks whether $\pi(w)$ belongs to the submonoid generated by $\pi(w_1), \ldots, \pi(w_n)$. Note that this problem generalizes the subgroup membership problem (or generalized word problem) for G.

The (uniform) rational subset membership problem for G takes as input a word $w \in \widetilde{\Sigma}^*$ and a finite automaton $\mathscr{A}$ over $\widetilde{\Sigma}^*$ and asks whether $\pi(w) \in \pi(L(\mathscr{A}))$. Recall that a finite automaton $\mathscr{A}$ over $\widetilde{\Sigma}$ is a tuple $\mathscr{A} = (Q, \widetilde{\Sigma}, \delta, q_0, F)$ where $\delta \subseteq Q \times \widetilde{\Sigma} \times Q$, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is the set of final states. The language $L(\mathscr{A})$ recognized by $\mathscr{A}$ is the set of all finite words from $\widetilde{\Sigma}^*$ that label a path from the initial state q_0 to some state from F. A set of the form $\pi(L(\mathscr{A}))$ is called a rational subset of G.

Study of rational subsets of groups began with the pioneering work of Benois on free groups [2] and Eilenberg and Schützenberger on abelian groups [6]. Rational subsets of groups were further studied in [1], where it was shown that a subgroup is a rational subset if and only if it is finitely generated, from which one obtains an easy proof of Howson's Theorem (in essence the same as Stallings' proof [22]). Recently, there has been renewed interest in the subject of rational subsets of groups [7, 9, 11, 13, 14, 17, 18], especially in connection to solving equations over groups [4, 5] and the isomorphism problem for toral relatively hyperbolic groups [3]. On the other hand, work of Stephen [23] and Ivanov, Margolis and Meakin [10] on the word problem for certain inverse monoids has spurred some interest in the submonoid membership problem. In particular, the word problem for one-relator inverse monoids motivated the paper [16],

where membership in positively generated submonoids of certain groups given by monoid presentations is considered. Recently, the authors have shown that the submonoid membership problem is undecidable in free metabelian groups of rank 2 [15].

Trivially, decidability of the rational subset membership problem implies decidability of the submonoid membership problem. In previous work [14], the authors showed that these two problems are recursively equivalent for graph groups (also known as right-angled Artin groups or free partially commutative groups) and gave the precise class of graph groups for which these problems are decidable. The authors were also able to prove that these problems are recursively equivalent for certain amalgamated free products (including free products) and HNN extensions with finite edge groups. This led the authors to conjecture that the rational subset membership and submonoid membership problems are recursively equivalent for groups with two or more ends. The main result of this paper is to establish our conjecture.

Recall that if Γ is a locally finite graph, then the space of ends of Γ is the projective limit $\varprojlim \pi_0(\Gamma \setminus F)$ where F runs over all finite subgraphs of Γ and $\pi_0(X)$ is the set of connected components of X. If G is a group with finite generating set Σ, then the number of ends $e(G)$ of G is the cardinality of the space of ends of its Cayley graph Γ with respect to Σ; it is well known that this number depends only on G and not Σ. Moreover, it is a result of Hopf that $e(G)$ is either $0, 1, 2$ or is infinite. Clearly $e(G) = 0$ if and only if G is finite. It is well known that $e(G) = 2$ if and only if G is virtually cyclic. Stallings' famous Ends Theorem [20, 21] says that $e(G) \geq 2$ if and only if G splits non-trivially as an amalgamated free product or an HNN extension over a finite subgroup, or, equivalently, G has an edge-transitive action without inversions on a simplicial tree with no global fixed points and finite edge stabilizers.

Our main theorem is then the following result.

Theorem 1. *Let G be a finitely generated group with two or more ends. Then the submonoid membership and rational subset membership problems for G are recursively equivalent. Moreover, there is a fixed rational subset of G with undecidable membership problem if and only if there is a fixed finitely generated submonoid of G with undecidable membership problem.*

We remark that the proof of the last statement in Theorem 1 is an existence argument: we do not give an algorithm that constructs a finitely generated submonoid with undecidable membership problem from a rational subset with undecidable membership problem.

The case that G in Theorem 1 has two ends is clear since a two-ended group is virtually cyclic and hence has a decidable rational subset membership problem. So the interesting case is a group with infinitely many ends.

The paper is organized as follows. Section 2 proves some results about groups acting on trees that will allow us to reduce Theorem 1 to a special case. This special case is then handled in Section 3.

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2   Groups acting on trees

The goal of this section is to reduce our considerations to a particular type of HNN extension. The notion of a (simplicial) tree is understood in the sense of Serre [19]. An edge e with initial vertex v and terminal vertex w is written as $v \xrightarrow{\ e\ } w$. Then, there is an inverse edge $w \xrightarrow{\ e^{-1}\ } v$, which never equals e. The pair $\{e, e^{-1}\}$ is called a *geometric edge*, as it corresponds to an edge of the geometric realization of the tree. Often we do not distinguish between an edge and the corresponding geometric edge. Let T be a tree. If v, w are vertices of a tree T, then $[v, w]$ denotes the geodesic (or reduced) path from v to w in T. Note that a path in T is geodesic if and only if it does not contain any backtracking, i.e, an edge followed by its inverse edge. An automorphism g of T is *without inversions* if, for each edge e of T, $ge \neq e^{-1}$. Let G be a group acting on T (by automorphisms of T). Then G acts *without inversions on T* if every $g \in G$ is without inversions. Throughout this paper, we tacitly assume that all actions of groups on trees are without inversions. We say that G acts *edge-transitively* on T if G acts transitively on the set of geometric edges. The stabilizer in G of a vertex or edge x of T will be denoted G_x. Of course, $G_e = G_{e^{-1}}$ for any edge e. A vertex v is called a *global fixed point* if $G = G_v$. The following lemma is well known, but we include a proof for completeness.

Lemma 2. *Let G be a finitely generated group acting edge-transitively on a tree T without global fixed points and with finite edge stabilizers (thus G has more than one end). Then the kernel of the action of G on T is the unique maximal finite normal subgroup of G.*

Proof. Let G_e be an edge stabilizer; it is finite by assumption. The kernel X of the action of G on T is clearly contained in G_e and hence is a finite normal subgroup. Let us show that every finite normal subgroup N of G is contained in X. Since N is finite, it fixes some vertex v of T [19, p. 36]. Since G has no global fixed point, we can find an element $g \in G$ such that $gv \neq v$. Let $[v, gv]$ be the geodesic from v to gv in T. Then since $N = gNg^{-1}$ fixes v and gv, it fixes $[v, gv]$ (since T is a tree) and hence it fixes some edge of this geodesic. Since G acts edge-transitively on T and N is normal, it follows that N fixes every edge of T and hence $N \subseteq X$. This completes the proof.   □

The chief result in this section is a reduction to the case of an HNN extension of a very special sort. Recall that if H is a group and $\varphi \colon A \to B$ is a *partial automorphism* of H, i.e., an isomorphism between subgroups A and B of H, then the corresponding HNN extension is the group $*_\varphi H$ given by the presentation $\langle H, t \mid t^{-1} a t = \varphi(a)\ (a \in A) \rangle$.

We first need some results concerning groups acting on trees, which may be of interest in their own right. We recall a result of Tits (cf. [19, p. 63]) that an automorphism g of a tree T (without inversions) either fixes a vertex, in which case it is called *elliptic*, or it leaves invariant a unique line T_g, called its *axis*, on which it acts via translation by some positive integer length. Elements of

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the latter sort are called *hyperbolic*. Of course, hyperbolic elements have infinite order.

Our next lemma is a geometric version of a result proved combinatorially (and separately) for amalgamated free products and HNN extensions with finite edge groups [12].

Lemma 3. *Let G be a finitely generated group acting edge-transitively on a tree T without global fixed points and with finite edge stabilizers (thus G has more than one end). Let $v \xrightarrow{e} w$ be an edge of T and suppose that G_e is finite proper subgroup of both G_v and G_w. Then there is an element $s \in G$ such that $G_e \cap sG_es^{-1} = X$ where X is the largest finite normal subgroup of G.*

Proof. Since the pointwise stabilizer of a geodesic is the intersection of the stabilizers of all its edges (and hence has size bounded by $|G_e|$), it follows that there is a non-empty finite geodesic path p whose pointwise stabilizer H has smallest cardinality amongst all geodesics in T. Without loss of generality, we may assume that p starts with either e or e^{-1} (else translate it to do so). Moreover, if p starts with e^{-1} we can replace e by e^{-1} and hence we may assume that p starts with e. Let $(se)^\varepsilon$ with $s \in G$ and $\varepsilon = \pm 1$ be the last edge traversed by p. We claim that we can choose p so that $\varepsilon = 1$. If this is not the case, i.e., the last vertex of p is sv, then we can choose $g \in G_{sv} \setminus G_{se}$ and consider the path $q = p(gse)$. We then have the situation in Figure 1. Any element of G that
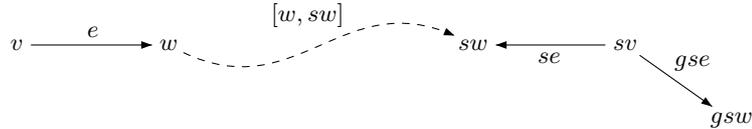

Fig. 1. When sv is the last vertex of p.

fixes q must then fix p and hence the pointwise stabilizer of q coincides with H by minimality. Thus replacing s by gs, we are in the desired situation. In particular, it follows that we may take s to be a hyperbolic element with $[v, sv]$ a fundamental domain for the action of s on its axis T_s, as the edges e, se are coherent in the sense of [19, p. 62]. Since every element that stabilizes e and se stabilizes p, we have $H = G_e \cap sG_es^{-1}$. We will show that $H = X$, where X is the largest finite normal subgroup of G. Recall from Lemma 2 that X is the kernel of the action of G on T. Hence, $X \subseteq H$. It remains to show that $H \subseteq X$, i.e., that H stabilizes every edge of T.

First we show that H is the pointwise stabilizer of T_s. Clearly, the pointwise stabilizer of T_s is contained in H, since p is contained in T_s. If the pointwise stabilizer of T_s is not equal to H, then some edge f of T_s is not stabilized by some element of H. By extending p to a reduced path in the line T_s containing f, we obtain a reduced path with a smaller pointwise stabilizer than p, a contradiction.

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Next, observe that H is closed under conjugation by all powers of s since T_s is $\langle s \rangle$-invariant and H is the pointwise stabilizer of T_s. Suppose now that ge is an edge of $T \setminus T_s$ with $g \in G$. Let r be the geodesic path from ge to T_s. Clearly, H stabilizes ge if and only if it stabilizes $s^n ge$ for some n since $s^n H s^{-n} = H$. Moreover, the geodesic path from $s^n ge$ to T_s is $s^n r$. Thus without loss of generality, we may assume that r ends at a vertex u of $s^m[v, sv]$ with $m > 2$. Then $p[sw, u]r^{-1}(ge)^\varepsilon$ is a reduced path (see Figure 2) containing p as an initial segment, for an appropriate choice of $\varepsilon = \pm 1$, and hence has pointwise stabilizer contained in H. But by choice of p, the pointwise stabilizer of $p[sw, u]r^{-1}(ge)^\varepsilon$ cannot be properly contained in H. Thus H stabilizes ge, i.e, H stabilizes every edge of T and hence $H = X$. This completes the proof. $\qquad\square$
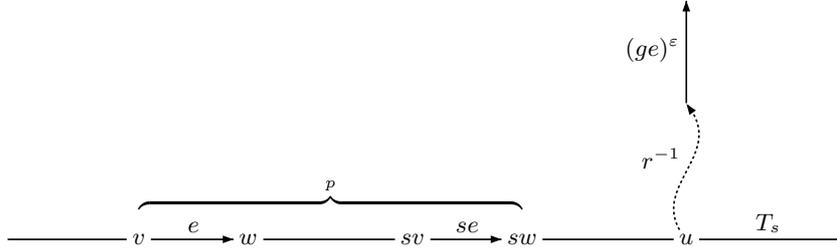
Fig. 2. The path $p[sw, u]r^{-1}(ge)^\varepsilon$.

Our next proposition uses the element s constructed above to create subgroups of a special form.

Proposition 4. *Let G be a group acting on a tree T. Let $v \xrightarrow{e} w$ be an edge of T and put $N = \bigcap_{g \in G} g G_e g^{-1}$. Suppose that:*

(a) $[G_v : G_e] \geq 3$;
(b) $[G_w : G_e] \geq 2$;
(c) There is an element $s \in G$ with $s G_e s^{-1} \cap G_e = N$.

Then G contains a subgroup isomorphic to an amalgamated free product of the form $G_v *_N (N \rtimes \mathbb{Z})$. Moreover, if G_e is finite, then G contains a subgroup isomorphic to $G_v *_N (N \times \mathbb{Z}) \cong *_{1_N} G_v$, where 1_N is the identity map on N.

Proof. We begin by observing that N consists of those element of G that stabilize all translates of e. Our third hypothesis (c) then says that there is a translate se so that the subgroup of elements fixing both e and se is exactly N. Our initial goal is to show that we may assume that the geodesic $[v, sv]$ from v to sv contains both e and se. First suppose the geodesic contains exactly one of these

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edges. Replacing e by se and s by s^{-1} if necessary, we may assume without loss of generality that it contains e. Since $G_{sw} \supsetneq G_{se}$ by (b), we can choose $g \in G_{sw} \setminus G_{se}$. Then we have the situation in Figure 3. Any element of G that
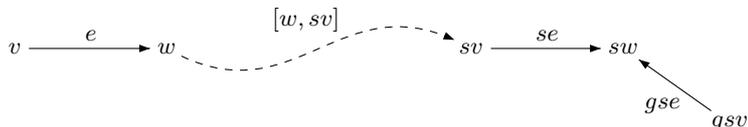
Fig. 3. When only e is in $[v, sv]$.

fixes e and gse must then fix e and se and so $gsG_es^{-1}g^{-1} \cap G_e = N$. Moreover, e and gse belong to $[v, gsv]$. Thus replacing s by gs, we are in the desired situation.

Next suppose that neither e nor se belong to $[v, sv]$. Choose $g \in G_{sw} \setminus G_{se}$. Then we have the picture in Figure 4. Again any element of G which fixes e
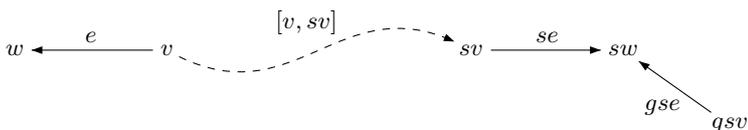

Fig. 4. When neither e nor se is in $[v, sv]$.

and gse must fix also se and hence belong to N. Thus $gsG_es^{-1}g^{-1} \cap G_e = N$. Moreover, gse is an edge of $[v, gsv]$ and so replacing s by gs leads us to the previous case where exactly one of the edges is on the geodesic.

Thus we may now assume that e and se belong to $[v, sv]$. Choose $g \in G_v \setminus G_e$. Then we have the situation in Figure 5. The edges e and sge are coherent in the sense of [19, p. 62]. Thus $x = sg$ is a hyperbolic element with axis $T_x = \langle x \rangle [v, sv]$ and $[v, sv]$ is a fundamental domain for the action of x on T_x (cf. [19, Sec. I.6]). Let X_1 be the connected component of $T \setminus \{se\}$ containing sv and let X_2 be the connected component of $T \setminus \{se\}$ containing sw.

Since $[G_v : G_e] \geq 3$ by (a) and T_x contains only one other (geometric) edge incident on v besides e, we can find $h \in G_v \setminus G_e$ so that $he \notin T_x$. Then $y = hxh^{-1}$ is a hyperbolic element with axis $T_y = hT_x$. The axes T_y and T_x intersect at the vertex $v = hv$, but do not coincide since $he \in T_y \setminus T_x$. Thus $T_y \cap T_x$ is either a point, a segment or an infinite ray. Notice that $X_1 \cap T_x$ and $X_2 \cap T_x$ each contain exactly one of the two ends of T_x. Interchanging e with se and s with s^{-1} in the statement of the proposition if necessary, we may assume that $T_y \cap T_x$ does
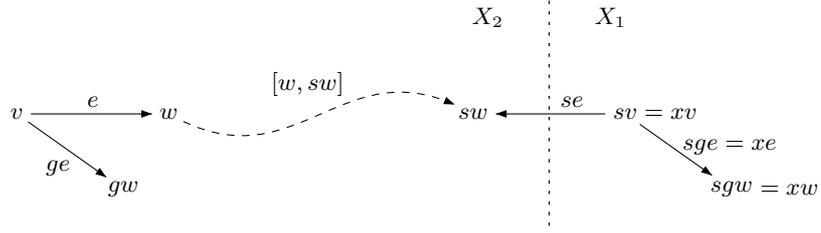
Fig. 5. A part of the axis of $x = sg$

not contain the end of T_x determined by $X_2 \cap T_x$. Then, since x acts on T_x by translations in the direction v to sv, by choosing $n > 0$ large enough, we can guarantee that $x^n T_y \cap T_x$ is contained in X_1 and does not contain the vertex sv. Putting $u = x^n y x^{-n}$ yields a hyperbolic element $u \in G$ with axis $T_u = x^n T_y$ such that $T_u \cap T_x \neq \emptyset$ is contained in X_1 and does not contain sv. The picture is either as in Figure 6 or 7.
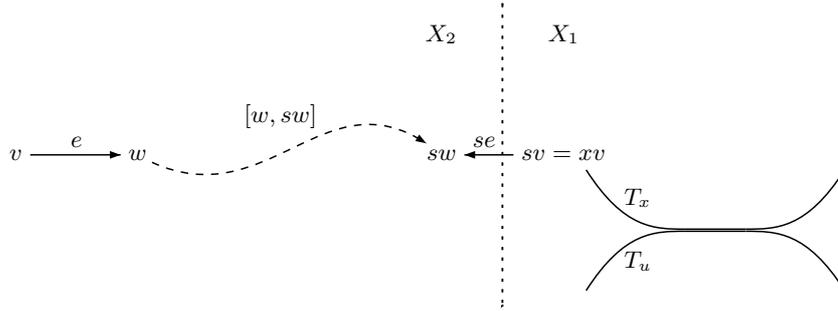
Fig. 6. The axes T_x and T_u have bounded intersection.

Let $H_1 = \langle N, u \rangle$ and $H_2 = G_v$. Since u is hyperbolic and elements of N are elliptic, it follows that $\langle u \rangle \cap N = \{1\}$. Consequently, $H_1 \cong N \rtimes \mathbb{Z}$ via the conjugation action of u on N. Also, if N is finite, then we can find a power $n > 0$ so that u^n acts trivially on N by conjugation. Then u^n is a hyperbolic element with the same axis as u, but a larger translation length. Replacing u by u^n, we may then assume that u commutes elementwise with N and so $H_1 \cong N \times \mathbb{Z}$. Also note that $H_1 \cap H_2 = N$. Indeed, if $h \in H_1$, we can write $h = u^n a$ with $a \in N$. Then since $a \in G_v$, we have $h \in G_v$ if and only if $u^n \in G_v$, if and only if $n = 0$ since u is hyperbolic.

We claim that H_1 and H_2 generate a subgroup isomorphic to $H_1 *_N H_2$. To prove this, we use the so-called Ping Pong Lemma [8]. The Ping Pong Lemma has the following setup: a group G acting on a set X, subgroups H_1 and H_2 of

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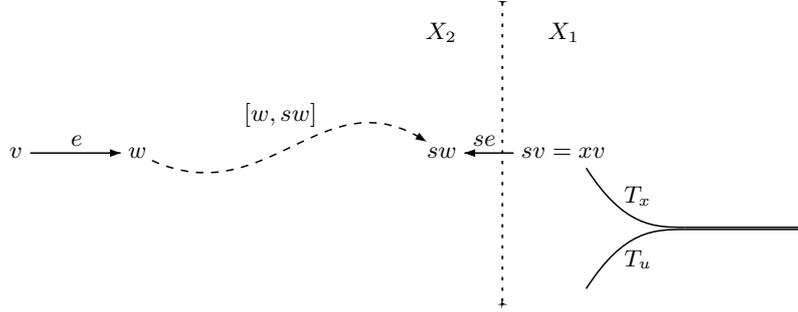
Fig. 7. The axes T_x and T_u have unbounded intersection.

G with $[H_1 : N] \geq 3, [H_2 : N] \geq 2$, where $N = H_1 \cap H_2$, and disjoint non-empty subsets $X_1, X_2 \subseteq X$ such that:

- $h_1 X_2 \subseteq X_1$ for all $h_1 \in H_1 \setminus N$;
- $h_2 X_1 \subseteq X_2$ for all $h_2 \in H_2 \setminus N$.

The conclusion of the Ping Pong Lemma is that the subgroup of G generated by H_1 and H_2 is isomorphic to the amalgamated free product $H_1 *_N H_2$. We apply the Ping Pong Lemma to our subgroups H_1, H_2. We take X to be T, whereas X_1 and X_2 have already been defined above. One should think of the edge se as the net of the ping pong table X. Note that in our case $[H_1 : N] = \infty$, whereas $[H_2 : N] \geq [G_v : G_e] \geq 3$.

So, let us show the two premises from the Ping Pong Lemma. For the first premise, let $k = u^m a \in H_1 \setminus N$ with $a \in N$. Then a fixes $[v, sv]$ because it fixes e and se. Let q be the geodesic from sv to $T_u \cap T_x$; it is non-empty by choice of u. As any non-trivial element u^m of $\langle u \rangle$ acts as a translation on the axis T_u, it follows that the geodesic $p = [ksv, sv] = [u^m sv, sv]$ must pass through a vertex of $T_u \cap T_x$ as per Figure 8. Since $T_u \cap T_x$ is contained in X_1, it follows that p is
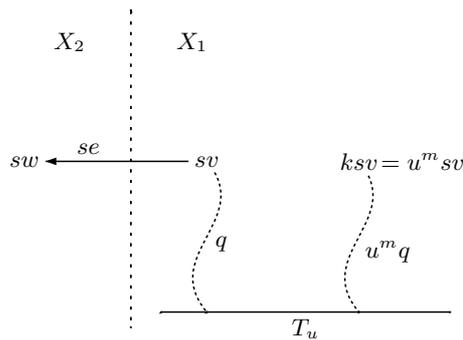
Fig. 8. The geodesic $p = [ksv, sv] = [u^m sv, sv]$

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contained in X_1. Let $v_0 \in X_2$. We claim that $k[v_0, sv] \cup [ksv, sv] = [kv_0, ksv] \cup p$ is the geodesic from kv_0 to sv, which implies that this path (and hence kv_0) is contained in X_1 since p is contained in X_1. To show that $k[v_0, sv] \cup [ksv, sv]$ is a geodesic, note that $k[v_0, sv]$ ends with the edge $(kse)^{-1}$. Hence, it suffices to show that p does not begin with the edge kse. The geodesic $p = [u^m sv, sv]$ begins with the path $u^m q$, see Figure 8. Since $q \subseteq X_1$ does not contain se, $u^m q$ does not contain $kse = u^m se$. We have thus shown $kX_2 \subseteq X_1$.

Finally, if $k \in H_2 \setminus N$, then k stabilizes v but not se (since if it stabilizes v and se, then it must also stabilize e and so belong to N by assumption on s). Since $kse \neq se$ and $kv = v$, there is a last vertex v_0 on the geodesic $[v, sv]$ fixed by k and $v_0 \neq sv$. Let $v_1 \in X_1$. Then by definition of X_1, it follows that $[v_1, v_0] = [v_1, sv] \cup [sv, v_0]$ and hence $[kv_1, v_0] = k[v_1, v_0] = k[v_1, sv] \cup k[sv, v_0]$. Notice that $k[sv, v_0] = [ksv, v_0]$ cannot contain any edge of $[v, sv]$ by choice of v_0. Therefore, $k[v_1, sv] \cup k[sv, v_0] \cup [v_0, sv] = [kv_1, sv]$ (see Figure 9) and hence $[kv_1, sv]$ contains the edge se. It follows that $kv_1 \in X_2$ and so $kX_1 \subseteq X_2$. The Ping Pong Lemma now yields that $\langle H_1, H_2 \rangle \cong H_1 *_N H_2$, completing the proof. $\square$
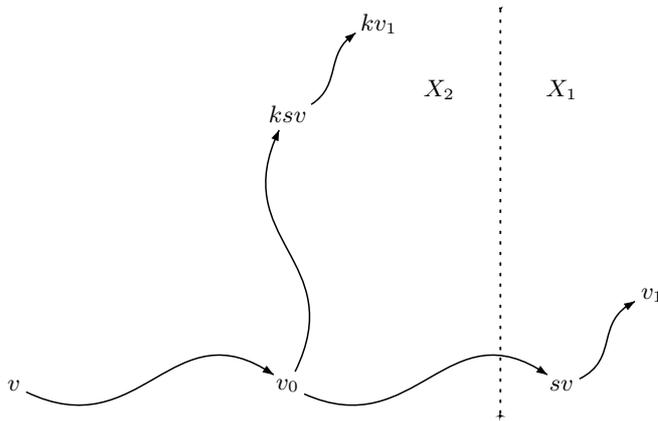
Fig. 9. The inclusion $kX_1 \subseteq X_2$.

The proposition admits the following corollary.

Corollary 5. *Let G be a finitely generated group splitting non-trivially over a finite subgroup A. Suppose that H is an infinite vertex group of the corresponding splitting. Then G has a subgroup isomorphic to the HNN extension $*_{1_X} H$ where X is the largest finite normal subgroup of G and $1_X \colon X \to X$ is the identity mapping.*

Proof. Let T be the Bass-Serre tree associated to the splitting of G over A. It follows that there is an edge $v \xrightarrow{e} w$ of T so that $G_e = A$ and $G_v = H$.

Moreover, since the splitting is non-trivial and also G cannot be an ascending HNN extension of H as H is infinite and A is finite, we must have $G_e \subsetneq G_w$. The action of G on T is edge-transitive and has no global fixed point. Lemma 3 provides $s \in S$ with

$$sG_es^{-1} \cap G_e = sAs^{-1} \cap A = X.$$

Observing that $X = \bigcap_{g \in G} gG_eg^{-1}$, Proposition 4 then allows us to conclude that G contains a subgroup isomorphic to $H *_X (X \times \mathbb{Z})$, which is easily seen to be isomorphic to the desired HNN extension. $\qquad\square$

The following technical result, which is the heart of this paper, will be proved in the next section.

Lemma 6. *Let $G = *_{1_X} H$ be an HNN extension with X a finite proper normal subgroup of H. Then the rational subset membership problem for H can be reduced to the submonoid membership problem for G. Moreover, if H has a fixed rational subset with undecidable membership problem, then G has a fixed finitely generated submonoid with undecidable membership problem.*

Let us now prove Theorem 1 assuming Lemma 6.

Proof of Theorem 1 For the first statement of Theorem 1, the non-trivial direction is to show that the rational subset membership problem for G reduces to the submonoid membership problem for G. So assume that G has decidable submonoid membership problem. We show that G has decidable rational subset membership problem.

By Stallings' Ends Theorem [20, 21] the group G splits non-trivially over a finite subgroup A. By the results of [11, 13], the rational subset membership problem for G reduces to the rational subset membership problem for the vertex group(s) of the splitting. Hence, it suffices to show that if H is a vertex group of the splitting, then H has decidable rational subset membership problem. If H is finite, there is nothing to prove, so we may assume that H is infinite. Then by Corollary 5, G contains a subgroup isomorphic to $*_{1_X} H$. Clearly, also $*_{1_X} H$ has a decidable submonoid membership problem. Lemma 6 implies that H has decidable rational subset membership problem.

The non-trivial direction of the second statement is to show that if G has a fixed rational subset with undecidable membership problem, then it has a fixed finitely generated submonoid with undecidable membership problem. So assume that G has a fixed rational subset with undecidable membership problem. In [13] the membership problem for a fixed rational subset of G is reduced to the membership problem of a finite number of fixed rational subsets of the vertex groups (in the splitting over A). Hence one of the vertex groups H has a fixed rational subset with undecidable membership problem; necessarily H is infinite. Corollary 5 then applies to allow us to deduce that G has a fixed subgroup of the form $*_{1_X} H$ where X is a finite proper normal subgroup of H and H has a fixed rational subset with undecidable membership problem.

Thus by Lemma 6, we conclude G has a fixed finitely generated submonoid with undecidable membership problem. This completes the proof. $\square$

3 The Proof of Lemma 6

Let us fix an HNN-extension $G = \langle H, t \mid t^{-1}xt = x \ (x \in X) \rangle$, where X is a proper finite normal subgroup of the finitely generated group H. Later, we will assume that G has decidable submonoid membership problem. We can assume that H is infinite, because otherwise H has decidable rational subset membership problem and the statement of Lemma 6 trivially holds. The group H acts on X by conjugation and since X is finite, the kernel N of this action has finite index in H (and hence is finitely generated). In particular, N is infinite and so $N \cap X$ is a proper subgroup of N. Since decidability of rational subset membership is a virtual property [9], to show that H has decidable rational subset membership, it suffices to prove that N has decidable rational subset membership.

Let $K = \langle N, t \rangle \leq G$. Then K centralizes X and so in particular, $K \cap X$ is contained in the center of K. Let us fix a finite group generating set Σ for N and denote by $\pi \colon \widetilde{\Sigma}^* \to N$ the canonical projection. Without loss of generality assume that $N \cap X \subseteq \widetilde{\Sigma}$. Suppose that $\mathscr{A} = (Q, \widetilde{\Sigma}, \delta, q_0, F)$ is a finite automaton, For a state $q \in Q$, let $X(q, \mathscr{A})$ be the set of all elements from $X \cap N$ represented by a word labeling a loop at state q. Note that these elements form a submonoid of $X \cap N$ and therefore (since $X \cap N$ is finite) a subgroup of $X \cap N$. Let $X(\mathscr{A}) \subseteq \widetilde{\Sigma}$ be the set $\bigcup_{q \in Q} X(q, \mathscr{A})$.

The proof of the following lemma is quite similar to the proof of Lemma 11 and Theorem 7 in [14].

Lemma 7. *From a given finite automaton $\mathscr{A}$ over $\widetilde{\Sigma}$ and an element $h \in N$ (given as a word over $\widetilde{\Sigma}$) we can construct effectively a finite subset $\Delta \subseteq K$ and an element $g \in K$ such that $h \in \pi(X(\mathscr{A})^* L(\mathscr{A}))$ if and only if $g \in \Delta^*$.*

Proof. Let $\mathscr{A} = (Q, \widetilde{\Sigma}, \delta, q_0, F)$. Without loss of generality assume that $Q \subseteq \mathbb{N}$ and $q \geq 1$ for all $q \in Q$. By introducing ε-transitions, we may also assume that the set of final states F consists of a single state q_f. We will construct a finite subset $\Delta \subseteq K$ and an element $g \in K$ such that $h \in \pi(X(\mathscr{A})^* L(\mathscr{A}))$ if and only if $g \in \Delta^*$.

Fix an element $k \in N \setminus X$ (hence also $k^{-1} \in N \setminus X$). Without loss of generality we may assume that $k \in \Sigma$. For every $q \in Q \subseteq \mathbb{N}$, define $[q] \in K$ by

$$[q] = t^q k t^{-q}.$$

Note that $[q]x = x[q]$ for all $x \in X$ since K centralizes X. Let

$$\Delta = \{[q]c[p]^{-1} \mid (q, c, p) \in \delta\} \quad \text{and} \quad g = [q_0]h[q_f]^{-1}. \tag{1}$$

where we abuse notation by treating $\widetilde{\Sigma}$ as if it were a subset of N. Observe that in (1), we have $c \in \widetilde{\Sigma} \cup \{1\} \subseteq N$, since we introduced ε-transitions. Also $\Delta \subseteq K$.

Note that $X(\mathscr{A})^*$ is contained in Δ^*. Indeed, if $x \in X(q, \mathscr{A})$, then we can write $x = a_1 \cdots a_m$ where there are transitions

$$(q, a_1, q_1), \ldots, (q_{m-1}, a_m, q) \in \delta.$$

Then, since x commutes with $[q]$, we have

$$x = [q]x[q]^{-1} = [q]a_1[q_1]^{-1}[q_1]a_2[q_2]^{-1} \cdots [q_{m-1}]a_m[q]^{-1} \in \Delta^*.$$

It follows that $X(\mathscr{A})^* \subseteq \Delta^*$.

We claim that $h \in \pi(X(\mathscr{A})^* L(\mathscr{A}))$ if and only if $g \in \Delta^*$. Let $\Gamma = \Sigma \cup \{t\}$. Let us define a X-*cycle* to be word in $\widetilde{\Gamma}^*$ of the form

$$[q_1]h_1[q_2]^{-1}[q_2]h_2[q_3]^{-1} \cdots [q_{\ell-1}]h_{\ell-1}[q_\ell]^{-1}[q_\ell]h_\ell[q_1]^{-1}$$

such that $\ell \geq 1$, $q_1, \ldots, q_\ell \in Q$, $h_1, \ldots, h_\ell \in \widetilde{\Sigma}^*$, and $h_1 \cdots h_\ell$ represents an element from the subgroup X. Since all elements from X commute with all $[q]$ ($q \in Q$), an X-cycle equals an element from X in G (actually an element of $N \cap X$).

A word in $\widetilde{\Gamma}^*$ of the form

$$[q_1]h_1[p_1]^{-1} \cdots [q_m]h_m[p_m]^{-1},$$

where $q_1, p_1, \ldots, q_m, p_m \in Q$ and $h_1, \ldots, h_m \in \widetilde{\Sigma}^*$, is called X-*cycle-free* if it does not contain an X-cycle as a factor.

Claim 1. Suppose that $p_1, \ldots, p_m, q_1, \ldots, q_m \in Q$ with $p_i \neq q_i$ ($1 \leq i \leq m$) and $q_i \neq p_{i+1}$ ($1 \leq i < m$). Then the element

$$g = [p_1]^{-1}[q_1][p_2]^{-1}[q_2] \cdots [p_m]^{-1}[q_m] \tag{2}$$

has a reduced expression in the HNN extension G starting with t and ending with t^{-1}.

Proof of Claim 1. We have

$$\prod_{i=1}^{m}[p_i]^{-1}[q_i] = \prod_{i=1}^{m} t^{p_i}k^{-1}t^{-p_i}t^{q_i}kt^{-q_i}$$

$$= \prod_{i=1}^{m} t^{p_i}k^{-1}t^{q_i-p_i}kt^{-q_i}$$

$$= t^{p_1}\left(\prod_{i=1}^{m-1} k^{-1}t^{q_i-p_i}kt^{p_{i+1}-q_i}\right)k^{-1}t^{q_m-p_m}kt^{-q_m}.$$

The latter word is a reduced word for g of the required form. This establishes Claim 1.

Let Y be the set of elements of the form (2). It follows immediately from the claim that any element g' of the form $h_0 y_1 h_1 y_2 h_2 \cdots y_r h_r$ with $r \geq 1$, $y_1, \ldots, y_r \in Y$,

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$h_1, \ldots, h_{r-1} \in N \setminus X$, and $h_0, h_r \in (N \setminus X) \cup \{1\}$ has a reduced expression for the HNN extension G containing t and t^{-1} and hence does not belong to N. Such elements g' will be called *good*. I.e., g' is good if it can be written as an alternating word in elements of Y and $N \setminus X$ with at least one factor from Y.

Claim 2. Assume that in the HNN-extension G

$$[n]h[r]^{-1} = [q_1]h_1[p_1]^{-1} \cdots [q_m]h_m[p_m]^{-1} \tag{3}$$

where $n, r, q_1, p_1, \ldots, q_m, p_m \in Q$, $h, h_1, \ldots, h_m \in \widetilde{\Sigma}^*$. If $\prod_{i=1}^m [q_i]h_i[p_i]^{-1}$ is X-cycle-free, then either $m = 0$, $n = r$, and $h = 1$, or all of the following hold:

 - $m \geq 1$ and $p_i = q_{i+1}$ for all $1 \leq i < m$,
 - $q_1 = n$ and $p_m = r$, and
 - $h = h_1 \cdots h_m$ in N.

Proof of Claim 2. We prove Claim 2 by induction over m. If $m = 0$, then (3) becomes

$$h = [n]^{-1}[r].$$

Since $h \in N$, Claim 1 immediately yields $n = r$, and hence $h = 1$, as required.

Now assume that $m \geq 1$.

Case 1. $m \geq 2$ and there is $1 \leq \ell < m$ such that $p_\ell = q_{\ell+1}$. Then (3) implies

$$[n]h[r]^{-1} = \left(\prod_{i=1}^{\ell-1} [q_i]h_i[p_i]^{-1} \right) [q_\ell](h_\ell h_{\ell+1})[p_{\ell+1}]^{-1} \left(\prod_{i=\ell+2}^m [q_i]h_i[p_i]^{-1} \right)$$

in G. Since $\prod_{i=1}^m [q_i]h_i[p_i]^{-1}$ from (3) is X-cycle-free, also the right hand side of this equality is X-cycle-free. Hence, we can apply the induction hypothesis and obtain:

 - $p_i = q_{i+1}$ for all $1 \leq i \leq \ell - 1$ and all $\ell + 1 \leq i < m$,
 - $q_1 = n$ and $p_m = r$, and
 - $h = h_1 \cdots h_m$ in N.

Since $p_\ell = q_{\ell+1}$ we obtain $p_i = q_{i+1}$ for all $1 \leq i < m$.

Case 2. $p_i \neq q_{i+1}$ for all $1 \leq i < m$. We have to show that $m = 1$, $q_1 = n$, $p_m = r$, and $h = h_1$ in N. Choose a set T of coset representatives for $N/(N \cap X)$ with $1 \in T$ and write $h_i = x_i h_i'$ with $h_i' \in T$ and $x_i \in N \cap X$, for $1 \leq i \leq m$. From this definition we get $h_1 h_2 \cdots h_m = x h_1' h_2' \cdots h_m'$ where $x = x_1 \cdots x_m \in N \cap X$ (since N centralizes X). Moreover, we also have

$$\prod_{i=1}^m [q_i]h_i[p_i]^{-1} = x \prod_{i=1}^m [q_i]h_i'[p_i]^{-1}$$

since $N \cap X$ is central in K.

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Thus, again using that x is central in K, we have

$$x^{-1}h = [n]^{-1}\left(\prod_{i=1}^{m}[q_i]h_i'[p_i]^{-1}\right)[r]. \tag{4}$$

Since the right hand side of (3) is X-cycle-free, if $h_i' = 1$ (i.e., $h_i \in X$), then $q_i \neq p_i$, for all $1 \leq i \leq m$ (else $[q_i]h_i[p_i]^{-1}$ is an X-cycle). Also we are assuming $p_i \neq q_{i+1}$. If $h_i' \neq 1$, then $h_i' \in N \setminus X$. Therefore, if both $n \neq q_1$ and $r \neq p_m$, then the right hand side of (4) represents a good element of G and so cannot belong to N, as was observed just before the statement of Claim 2. This contradicts $x^{-1}h \in N$. Similarly, suppose $n = q_1$ and $p_m \neq r$. Then

$$x^{-1}h = h_1'[p_1]^{-1}\left(\prod_{i=2}^{m}[q_i]h_i'[p_i]^{-1}\right)[r]$$

which is again good, a contradiction. The case $n \neq q_1$ and $p_m = r$, is handled analogously. If $n = q_1, p_m = r$ and $m \geq 2$, then

$$x^{-1}h = h_1'[p_1]^{-1}\left(\prod_{i=2}^{m-1}[q_i]h_i'[p_i]^{-1}\right)[q_m]h_m'$$

and so is again good, a contradiction. The only remaining case is when $n = p_1$, $r = p_m$ and $m = 1$. Then $[n]h[r]^{-1} = [n]h_1[r]^{-1}$ in G and so $h = h_1$ in N, as required. This proves Claim 2. An immediate consequence of Claim 2 is:

Claim 3. Assume that in the HNN-extension G

$$[q_0]h[q_f]^{-1} = [q_1]a_1[p_1]^{-1}\cdots[q_m]a_m[p_m]^{-1},$$

where $(q_i, a_i, p_i) \in \delta$ (i.e., $[q_i]a_i[p_i]^{-1} \in \Delta$) for $1 \leq i \leq m$. If $\prod_{i=1}^{m}[q_i]a_i[p_i]^{-1}$ is X-cycle-free, then $h \in \pi(L(\mathscr{A}))$.

The case $m \geq 1$ follows immediately from Claim 2. In case $m = 0$, Claim 2 implies $q_0 = q_f$ and $h = 1$. But $q_0 = q_f$ implies $\varepsilon \in L(\mathscr{A})$ and hence $h \in \pi(L(\mathscr{A}))$.

Now we are ready to prove that $h \in \pi(X(\mathscr{A})^*L(\mathscr{A}))$ if and only if $g = [q_0]h[q_f]^{-1} \in \Delta^*$. First assume that $h \in \pi(X(\mathscr{A})^*L(\mathscr{A}))$. Let $h = xa_1\cdots a_m$ in N, where $a_1\cdots a_m \in L(\mathscr{A})$ and $x \in X(\mathscr{A})^*$. Hence, there are states $q_0, \ldots, q_m$ with $(q_{i-1}, a_i, q_i) \in \delta$ for $1 \leq i \leq m$, and $q_m = q_f$. It was observed earlier that $X(\mathscr{A})^* \subseteq \Delta^*$. Moreover, since x commutes with $[q_0]$, we obtain in G:

$$g = [q_0]h[q_f]^{-1} = x[q_0]a_1a_2\cdots a_m[q_f]^{-1} = x\prod_{i=1}^{m}[q_{i-1}]a_i[q_i]^{-1} \in \Delta^*.$$

Next assume that $g = [q_0]h[q_f]^{-1} \in \Delta^*$. Thus,

$$[q_0]h[q_f]^{-1} = [q_1]a_1[p_1]^{-1}\cdots[q_m]a_m[p_m]^{-1}$$

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in G, where $q_1, p_1, \ldots, q_m, p_m \in Q$, $a_1, \ldots, a_m \in \widetilde{\Sigma} \cup \{\varepsilon\}$, and $(q_i, a_i, p_i) \in \delta$ for $1 \leq i \leq m$. Every X-cycle that occurs in $\prod_{i=1}^m [q_i] a_i [p_i]^{-1}$ can be reduced in G to an element from some subgroup of the form $X(q, \mathscr{A})$. Moreover, since X is central in K, we can move each of these elements to the beginning and then to the left-hand side $[q_0] h [q_f]^{-1}$ by taking inverses. Performing this reduction until no further X-cycles remain, we obtain in G an equality

$$[q_0] x^{-1} h [q_f]^{-1} = [n_1] b_1 [r_1]^{-1} [n_2] b_2 [r_2]^{-1} \cdots [n_\ell] b_\ell [r_\ell]^{-1} \tag{5}$$

where $(n_i, b_i, r_i) \in \delta$ and $x \in X(\mathscr{A})^*$. Since the right-hand side of (5) is X-cycle free, Claim 3 implies $x^{-1} h \in \pi(L(\mathscr{A}))$, i.e., $h \in \pi(X(\mathscr{A})^* L(\mathscr{A}))$. $\qquad\square$

Let us consider again a finite automaton $\mathscr{A} = (Q, \widetilde{\Sigma}, \delta, q_0, F)$ over the alphabet $\widetilde{\Sigma}$. A subset $P \subseteq Q$ is called *admissible* if $q_0 \in P$ and $P \cap F \neq \emptyset$. For every admissible subset $P \subseteq Q$ we define the automaton $\mathscr{A}_P$ as follows:

$$\mathscr{A}_P = (P \times \{R \mid R \subseteq P, q_0 \in R\}, \widetilde{\Sigma}, \delta_P, (q_0, \{q_0\}), \{(q, P) \mid q \in P \cap F\}),$$

where the transition relation δ_P is given by

$$\delta_P = \{((p, R), a, (q, R \cup \{q\})) \mid p, q \in P, (p, a, q) \in \delta\}.$$

The automaton $\mathscr{A}_P$ works as follows. The construction is such that there is a run in $\mathscr{A}_P$ of a word w from $(q_0, \{q_0\})$ to (p, R) if and only if there is a run from q_0 to p in $\mathscr{A}$ that visits precisely the states in R. In particular, $L(\mathscr{A}_P)$ consists of all words $w \in L(\mathscr{A})$ that label an accepting path in $\mathscr{A}$ using exactly the vertices from P.

Lemma 8. *For every finite automaton $\mathscr{A} = (Q, \widetilde{\Sigma}, \delta, q_0, F)$ and $h \in N$, we have: $h \in \pi(L(\mathscr{A})) \iff \exists P \subseteq Q$ admissible such that $h \in \pi(L(\mathscr{A}_P))$.*

Proof. Since $L(\mathscr{A}_P) \subseteq L(\mathscr{A})$ for all admissible subsets, to prove the lemma it suffices to observe that if $h = \pi(w)$ with $w \in L(\mathscr{A})$, then the set P of states visited by a successful run of w is admissible and $w \in L(\mathscr{A}_P)$. $\qquad\square$

We are now ready to prove Lemma 6.

Proof of Lemma 6 Assume that $G = *_{1_X} H$, where X is a proper finite normal subgroup of H, has a decidable submonoid membership problem. We shall exhibit an algorithm for the rational subset membership problem for H. If H is finite, there is nothing to prove. So assume that H is infinite. Let N be the centralizer of X in H and let $K = \langle N, t \rangle \leq G$. Then N has finite index in H and so it suffices to show that N has a decidable rational subset membership problem by [9]. For this, let $\mathscr{A} = (Q, \widetilde{\Sigma}, \delta, q_0, F)$ be a finite automaton and let $h \in N$. We show how to decide $h \in L(\mathscr{A})$. By Lemma 8 it suffices to decide whether $h \in \pi(L(\mathscr{A}_P))$ for each admissible subset $P \subseteq Q$. Let us abbreviate the automaton $\mathscr{A}_P$ by $\mathscr{B}$. Recall that the set of states of $\mathscr{B}$ is $P \times \{R \subseteq P \mid q_0 \in R\}$.

Now we apply Lemma 7 (with $\mathscr{B}$ in place of $\mathscr{A}$) and construct an element $g \in K$ and a finite subset $\Delta \subseteq K$ such that

$$h \in \pi(X(\mathscr{B})^*L(\mathscr{B})) \quad \Longleftrightarrow \quad g \in \Delta^*.$$

Hence it suffices to prove $\pi(L(\mathscr{B})) = \pi(X(\mathscr{B})^*L(\mathscr{B}))$.

It is immediate that $\pi(L(\mathscr{B})) \subseteq \pi(X(\mathscr{B})^*L(\mathscr{B}))$. So it remains to establish that $\pi(X(\mathscr{B})^*L(\mathscr{B})) \subseteq \pi(L(\mathscr{B}))$. To do this, it suffices to show that

$$\pi(xL(\mathscr{B})) \subseteq \pi(L(\mathscr{B}))$$

for every $x \in X(\mathscr{B})$ (then we can conclude by induction over the length of the word from $X(\mathscr{B})^*$). Let us take $x \in X((p, R), \mathscr{B})$ for some state (p, R) of $\mathscr{B}$ and consider a word $xa_1a_2 \cdots a_m$ with $a_1a_2 \cdots a_m \in L(\mathscr{B})$. Hence, there exist states $(q_0, R_0), \ldots, (q_m, R_m)$ of $\mathscr{B}$ such that:

- $R_0 = \{q_0\}$,
- $((q_i, R_i), a_{i+1}, (q_{i+1}, R_{i+1}))$ is a transition of $\mathscr{B}$ for $0 \le i \le m-1$, and
- $R_m = P$, $q_m \in P \cap F$.

Note that we have $P = \{q_0, \ldots, q_m\}$. Hence, there exists an j such that $q_j = p$ (recall that $p \in P$ was such that $x \in X((p, R), \mathscr{B})$). Since $X((p, R), \mathscr{B}) \subseteq N \cap X$ is central in N, we have

$$xa_1a_2 \cdots a_m = a_1 \cdots a_j x a_{j+1} \cdots a_m$$

in N. Then there is a loop at state $(p, R) = (q_j, R)$ labeled with $c_1 \cdots c_k$ and $x = \pi(c_1 \cdots c_k)$ by definition of $X((p, R), \mathscr{B})$. In particular, $c_1 \cdots c_k$ reads a loop in $\mathscr{A}$ at p visiting only states contained in $R \subseteq P$. Now, starting from the initial state (q_0, R_0) we can read the word $a_1 \cdots a_j c_1 \cdots c_k a_{j+1} \cdots a_m$ on the automaton $\mathscr{B}$. The state reached after reading $a_1 \cdots a_j c_1 \cdots c_k a_{j+1} \cdots a_\ell$, for $\ell \ge j$, is of the form (q_ℓ, S_ℓ), where $R_\ell \subseteq S_\ell$. Hence, $S_m = P$. This shows that $a_1 \cdots a_j c_1 \cdots c_k a_{j+1} \cdots a_m \in L(\mathscr{B})$. This completes the proof that N has decidable rational subset membership problem. Therefore, H has decidable rational subset membership problem.

On the other hand, if H has a fixed rational subset with undecidable membership problem, then it follows from the construction in [9] that N has a fixed rational subset $\pi(L(\mathscr{A}))$ with undecidable membership problem. It follows from Lemma 8 that $L(\mathscr{A}_P)$ has an undecidable membership problem for some P. Consequently, Lemma 7 provides a fixed finitely generated submonoid of G with undecidable membership problem.

This completes the proof of Lemma 6, thereby establishing Theorem 1. $\quad\square$

4 Concluding remarks and open problems

We have shown that, for every group with at least two ends, the rational subset membership problem and the submonoid membership problem are recursively

equivalent. Moreover, in a previous paper we proved that these two problems are recursively equivalent for graph groups as well [14].

It is easy to see that there exists a group G with infinitely many ends for which submonoid membership is undecidable but the generalized word problem (i.e., membership in finitely generated subgroups) is decidable. Take any group H for which the generalized word problem is decidable but submonoid membership is undecidable (e.g., the free metabelian group of rank 2 [15] or the graph group defined by a path with 4 nodes [14]). Then the same is true for $H * \mathbb{Z}$, which has infinitely many ends.

The obvious remaining open question is whether there exists a (necessarily one-ended) finitely generated group for which rational subset membership is undecidable but submonoid membership is decidable. We conjecture that such a group exists. We also conjecture that decidability of submonoid membership is *not* preserved by free products, because otherwise rational subset membership and submonoid membership would be equivalent for all finitely generated groups.

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