The Rational Subset Membership Problem for Groups: A Survey

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Abstract

The class of rational subsets of a group G is the smallest class that contains all finite subsets of G and that is closed with respect to union, product and taking the monoid generated by a set. The rational subset membership problem for a finitely generated group G is the decision problem, where for a given rational subset of G and a group element g it is asked whether $g \in G$. This paper presents a survey on known decidability and undecidability results for the rational subset membership problem for groups. The membership problems for finitely generated submonoids and finitely generated subgroups will be discussed as well.

1 Introduction

The study of algorithmic problems in group theory has a long tradition. Dehn, in his seminal paper from 1911 [13], introduced the word problem (Does a given word over the generators represent the identity?), the conjugacy problem (Are two given group elements conjugate?) and the isomorphism problem (Are two given finitely presented groups isomorphic?), see [38] for general references in combinatorial group theory. Starting with the work of Novikov and Boone from the 1950's, all three problems were shown to be undecidable for finitely presented groups in general. A generalization of the word problem is the *subgroup membership problem* (also known as the *generalized word problem*) for finitely generated groups: Given group elements $g, g_1, \dots, g_n$, does g belong to the subgroup generated by $g_1, \dots, g_n$? Explicitly, this problem was introduced by Mihailova in 1959 [42], although Nielsen had already presented an algorithm for the subgroup membership problem for free groups in his paper from 1921 [47].

Motivated partly by automata theory, the subgroup membership problem was further generalized to the *rational subset membership problem*. Assume that the group G is finitely generated by the set X (where $a \in X$ if and only if $a^{-1} \in X$). A finite automaton A with transitions labeled by elements of X defines a subset $L(A) \subseteq G$ in the natural way; such subsets are the rational subsets of G, see Section 2 and 3 for precise definitions. The rational subset membership problem asks whether a given group element belongs to $L(A)$ for a given finite automaton (in fact, this problem makes sense for any finitely generated monoid). The notion of a rational subset of a monoid can be traced back to the work of Eilenberg and Schützenberger from 1969 [15]. The first decidability result for the rational subset membership problem was shown by Benois [5]: Every finitely generated free group has a decidable rational subset membership problem.

It seems that after Benois' work the rational subset membership problem had been forgotten for a long time. Aspects of rational sets in monoids that are close to classical formal language theory were studied in the 1980s and 1990s, see [7, 18] for surveys. Only in 1999, Grunschlag returned to the rational subset membership problem in his thesis [19]. He proved that the rational subset membership problem is decidable for finitely generated abelian groups and that decidability of the rational subset membership problem is preserved by finite extensions. Also in 1999, Roman'kov presented at a conference a proof, showing that the rational subset membership problem is undecidable for nilpotent groups (even of class 2), see Section 7. The next step was done by Kambites, Silva, and Steinberg in 2006 [26]. They proved that the rational subset membership problem is decidable for the fundamental group of a graph of groups, provided that (i) all edge groups are finite and (ii) every vertex group has a decidable rational subset membership problem, see Section 5. Further (un)decidability results on the rational subset membership problem in various classes of groups (right-angled Artin groups, metabelian groups, wreath products) can be found in [33, 36, 37], see Sections 6, 8, and 9. The latter three papers also studied the submonoid membership problem, which sits in between the subgroup membership problem and the rational subset membership problem. The input consists of group elements $g, g_1, \dots, g_n \in G$ and it is asked whether g belongs to the submonoid of G

generated by $g, g_1, \ldots, g_n$. In [35] it was shown that if the group G has at least two ends, then the rational subset membership problem for G is decidable if and only if the submonoid membership problem for G is decidable, see Section 10.

Rational subsets of groups also found applications for the solution of word equations (here, quite often the term rational constraint is used) [14, 31]. In automata theory, rational subsets are tightly related to valence automata: A valence automaton over a monoid M (also the term M-automaton is sometimes used) is a finite automaton, where every transition is labeled with an input symbol and an element of M. A word w is accepted by such a valence automaton, if there exists a path from the initial state to a final state such that: (i) the concatenation of the inputs symbols along this path yields the word w and (ii) the product of the M-elements along the path is the monoid identity. For any group G, the emptiness problem for valence automata over G is decidable if and only if G has a decidable rational subset membership problem. See [10, 16, 24, 26, 60, 61] for details on valence automata.

2 Finite automata

We assume that the reader has some background on computability theory. She or he should be familiar with the concepts of a *decidable problem* (also called *computable problem*) and *undecidable problem* (also called *unsolvable problem* or *insoluble problem*), see e.g. [51] for background. In Section 4, we present a proof that requires some basic knowledge of complexity theory, in particular the theory of NP-completeness, see [48] for background. Although we give all needed definitions related to finite automata, some background on automata theory (see e.g. [21]) makes the paper certainly easier to read.

Let X be a finite set of symbols, which is also called an alphabet. With X^* we denote the set of all finite words $w = a_1 a_2 \cdots a_n$ with $a_1, \ldots, a_n \in X$. If $n = 0$, then w is the empty word, which is also denoted by ε. A subset of X^* is also called a *language*. A finite automaton over X is a tuple $\mathcal{A} = (Q, \Delta, q_0, F)$, where

- $\Delta \subseteq Q \times X \times Q$ is the set of transitions,
- $q_0 \in Q$ is the initial state, and
- $F \subseteq Q$ is the set of final states.

The language accepted by $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of all words $w = a_1a_2\cdots a_n \in X^*$ for which there exist states $q_1, q_2, \ldots, q_n \in Q$ such that $(q_{i-1}, a_i, q_i) \in \Delta$ for $1 \leq i \leq n$ (note that for $i = 1$, $q_{i-1} = q_0$ is the initial state) and $q_n \in F$. Languages of the form $L(\mathcal{A})$ for $\mathcal{A}$ a finite automaton are called *regular*.

A finite automaton over X with ε-transitions is defined as above, except that $\Delta \subseteq Q \times (X \cup \{\varepsilon\}) \times Q$. A transition $(q, \varepsilon, p) \in \Delta$ means that the automaton can move from state q to state p without reading an input symbol. It is well-known that for every finite automaton with ε-transitions there exists an ordinary finite automaton (without ε-transitions) that accepts the same language [21]. Allowing ε-transitions sometimes simplifies technical details in proofs.

3 Rational subsets of groups

We assume that the reader has some background in combinatorial group theory. A classical reference is [38]. Let G be a finitely generated group and X a finite symmetric generating set for G (symmetric means that X is closed under taking inverses). This mean that the canonical morphism $\pi : X^* \to G$ that maps a word $w \in X^*$ to the group element of G represented by w is surjective. Hence, elements of group G can be represented by finite words over the alphabet X. When we say below that the input for a decision problem consists of a group element $g \in G$ (plus possibly some other objects), then we actually mean that the input consists of a finite word $w \in X^*$ that represents the group element g.

Let us fix a monoid M. For a subset $B \subseteq M$ we denote with B^* the *submonoid* of M generated by B. Of course we have to distinguish B^* from the set of all words over B, which is also denoted by B^*. It will be always clear, whether B^* is viewed as the set of all words over B or as the submonoid of M generated by B. In case M is a group, we denote with $\langle B \rangle$ the subgroup generated by B. The set of *rational subsets* of M is the smallest subset of 2^M that (i) contains all finite subsets of M and (ii) that is closed under union, product, and *.

In the following, we will mainly consider rational subsets of a group G. If G is finitely generated by X^* and $\pi : X^* \to G$ is the corresponding canonical homomorphism, then rational subsets of G can be represented by finite automata over X. The following result can deduced from Kleene's theorem for regular languages, see [18] for a proof:

Proposition 1. *Let G be a finitely generated group, let X be a finite generating set for G, and let $\pi : X^* \to G$ be the corresponding canonical homomorphism. A subset $L \subseteq G$ is rational if and only if there is a finite automaton $\mathcal{A}$ over X such that $L = \pi(L(\mathcal{A}))$.*

This characterization of rational subsets is useful since it allows to represent a rational subset of G by a finite automaton over X.

We consider the following decision problem for a finitely generated group G together with a canonical morphism $\pi : X^* \to G$.

Decision problem 2. (Rational subset membership problem for G)

- *INPUT: A finite automaton $\mathcal{A}$ over X and an element $g \in G$.*
- *QUESTION: Does $g \in \pi(L(\mathcal{A}))$ hold?*

Note that $g \in L(\mathcal{A})$ if and only if $1 \in L(\mathcal{A})g^{-1}$. Moreover, the set $L(\mathcal{A})g^{-1}$ is rational too and a finite automaton for this set can be constructed from $\mathcal{A}$ and g. Hence, the rational subset membership problem for G is equivalent to the following problem:

- *INPUT: A finite automaton $\mathcal{A}$ over X.*
- *QUESTION: Does $1 \in \pi(L(\mathcal{A}))$ hold?*

Decision problem 3. (Submonoid membership problem for G)

- *INPUT: Elements $g, g_1, \ldots, g_n \in G$.*
- *QUESTION: Does $g \in \{g_1, \ldots, g_n\}^*$ hold?*

Decision problem 4. (Subgroup membership problem for G)

- *INPUT: Elements $g, g_1, \ldots, g_n \in G$.*
- *QUESTION: Does $g \in \langle g_1, \ldots, g_n \rangle$ hold?*

The subgroup membership problem for G is also known as the *generalized word problem for G* or as the *occurrence problem for G*.

Strictly speaking, we should speak of the rational subset membership problem for G with respect to $\pi : X^* \to G$, since another choice for the generating set leads to another decision problem. On the other hand, if X and Y are two finite generating sets for G with canonical morphisms $\pi : X^* \to G$ and $\sigma : Y^* \to G$, then the rational subset membership problem for G with respect to $\pi : X^* \to G$ is decidable, if and only if the rational subset membership problem for G with respect to $\sigma : Y^* \to G$ is decidable. For the proof, one chooses a morphism $\rho : Y^* \to X^*$ such that for every $a \in Y$, $\sigma(a) = \pi(\rho(a))$ (clearly, such a morphism exists). Then, for $w \in Y^*$ and a finite automaton $\mathcal{A}$ over Y, we have $\sigma(w) \in \sigma(L(\mathcal{A}))$ if and only if $\pi(\rho(w)) \in \pi(L(\mathcal{B}))$. Here, $\mathcal{B}$ is the automaton over X that results from $\mathcal{A}$ by replacing every a-labelled transition ($a \in Y$) by a chain of transitions that is labelled with the word $\rho(a)$. A similar remark applies to the submonoid membership problem and the subgroup membership problem for G.

Clearly, decidability of the rational subset membership problem for G implies decidability of the submonoid membership problem for G, and the latter implies decidablity of the subgroup membership problem for G.

Note that in the above three decision problems, the input consists of a group element g and a finite description of a subset $Z \subseteq G$, and it is asked whether $g \in Z$. A more restricted setting is obtained by fixing a subset $Z \subseteq G$. For this set Z, we can consider the following decision problem:

Decision problem 5. (Membership problem for the set $Z \subseteq G$)

- *INPUT: An element $g \in G$.*
- *QUESTION: Does $g \in Z$ hold?*

Open problem 6. *Is there a finitely generated group G with the following properties?*

- *For every rational subset $R \subseteq G$, the membership problem for R is decidable.*
- *The rational subset membership problem for G is undecidable.*

The same question can be considered for rational subsets replaced by finitely generated submonoids or finitely generated subgroups.

One should note that a positive answer to this problem is conceivable: There might exist a group G for which there is no algorithm that decides the rational subset membership problem for G, but for every rational subset

$R \subseteq G$ there is an algorithm A_R that checks whether a given group element belongs to R. These algorithms A_R must be completely unrelated in the sense that they do not follow a uniform scheme.

Of course, one may also generalize Problem 2 further, e.g. by considering context-free languages. Given a context-free grammar $\mathcal{G}$ over the symmetric generating set X of the group G and a group element $g \in G$, one can ask whether $g \in \pi(L(\mathcal{G}))$. But this problem is already undecidable for free groups: To see this, take a finitely presented group $G = \mathsf{Gp}\langle X \mid R\rangle$ with an undecidable word problem. Here $R \subseteq X^*$ is a finite set of relators. Then, for a given word $w \in X^*$ we have $w = 1$ in G if and only if in the free group $F(X)$, w belongs to the normal closure of R. But the latter is the canonical image of the context-free language $L = \{crc^{-1} \mid r \in R, c \in X^*\}$. Hence, if the word problem for G is undecidable, then the membership problem for the free group image of the context-free language L is undecidable.

By the last paragraph, the membership problem for (images of) context-free sets is already undecidable for the simplest finitely generated groups (namely free groups).[1] On the other hand, the following sections will show that for the rational subset membership problem we can prove non-trivial decidability results. This is one of the reasons for restricting the membership problem to rational sets in this paper.

4 Classical results

The first decidability result for the rational subset membership problem was shown by Benois in 1969 for free groups [5]:

Theorem 7. *Every free group of finite rank has a decidable rational subset membership problem.*

This result can be shown by a simple automata saturation procedure. Consider a free group $F(Y)$, where Y (a finite set) generates $F(Y)$ as a group. Let $X = Y \cup Y^{-1}$. Let $\mathcal{A} = (Q, \Delta, q_0, F)$ be a finite automaton with ε-transitions over the alphabet X. Since we will add ε-transitions to the automaton, will start with an automaton with ε-transitions from the very beginning. As remarked in the previous section, it suffices to check, whether $1 \in \pi(L(\mathcal{A}))$. For this we iterate the following operation as long

[1]The only class of groups with a decidable membership problem for context-free sets, the author is aware of, are finitely generated virtually abelian groups.

as possible: If there are transitions $(p, a, q), (r, a^{-1}, s) \in \Delta$ with $a \in X$, and state r can be reached from state q by a sequence of ε-transitions, then we add the ε-transition (p, ε, s) to Δ. The order in which we add ε-transitions is not important. Note that we only add new transitions but we do not add new states. Hence the saturation process has to terminate after at most $|Q|^2$ many steps. Let $\mathcal{B}$ be the resulting automaton with ε-transitions. Then, one can show the following:

- $\pi(L(\mathcal{A})) = \pi(L(\mathcal{B}))$ (this follows by induction on the construction of $\mathcal{B}$)
- If $w \in L(\mathcal{B})$ and w is of the form $w = uaa^{-1}v$ with $u, v \in X^*$ and $a \in X$, then also $uv \in L(\mathcal{B})$.

Hence, we have $1 \in \pi(L(\mathcal{A}))$ if and only if $1 \in \pi(L(\mathcal{B}))$ if and only if there is a word $w \in L(\mathcal{B})$ such that w can be reduced by cancellations of the form $aa^{-1} \to \varepsilon$ ($a \in X$). But the latter condition is equivalent to $\varepsilon \in L(\mathcal{B})$. Hence, $1 \in \pi(L(\mathcal{A}))$ if and only if $\varepsilon \in L(\mathcal{B})$, and the latter means that there is a path consisting only if ε-transitions leading from the initial state q_0 to a final state. This conditions can be checked by an algorithm.

It is worth mentioning that the above algorithm works in polynomial time, see [6] for a precise complexity analysis.

Next, let us consider finitely generated abelian groups. The following result was shown by Grunschlag in his thesis [19] using integer linear programming.

Theorem 8. *Every finitely generated abelian group has a decidable rational subset membership problem.*

Grunschlag reduces the rational subset membership problem for finitely generated abelian groups to integer linear programming, which is a classical NP-complete problem. It turns out that already the submonoid membership problem for free abelian groups $\mathbb{Z}^k$ is NP-complete if k is part of the input. To see this, we start with the NP-complete problem 1-in-3 SAT [48]. The input is a conjunction

$$\psi = \bigwedge_{i=1}^{m} C_i,$$

where every C_i is a disjunction of three literals (a literal is a boolean variable or a negated boolean variable). Let $x_1, \ldots, x_n$ be the boolean variables that appear in ψ and let $C_i = \tilde{x}_{i_1} \vee \tilde{x}_{i_2} \vee \tilde{x}_{i_3}$, where $\tilde{x}_{i_j} \in \{x_{i_j}, \neg x_{i_j}\}$. It is asked, whether there exists a truth assignment for the variables $x_1, \ldots, x_n$ such that in each disjunction C_i exactly one literal becomes true. This is true if and only if the following system of linear equations in the $2n$ variables $x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n$ has a solution in $\mathbb{N}$:

$$\begin{aligned} x_i + \overline{x}_i &= 1 \text{ for } 1 \leq i \leq n \\ \tilde{x}_{i_1} + \tilde{x}_{i_2} + \tilde{x}_{i_3} &= 1 \text{ for } 1 \leq i \leq m \end{aligned}$$

In the second equation, we identify the literal $\neg x_{i_j}$ with the variable $\overline{x}_{i_j}$. This system can be written as

$$\sum_{i=1}^{n} (x_i \cdot \mathbf{a}_i + \overline{x}_i \cdot \mathbf{b}_i) = \mathbf{c},$$

for $\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_n, \mathbf{b}_n, \mathbf{c} \in \mathbb{Z}^{n+m}$. This system is solvable in the natural numbers if and only if $\mathbf{c}$ belongs to the submonoid generated by $\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_n, \mathbf{b}_n$.

Note that in the above NP-hardness proof we have to assume that the dimension (which is $n + m$) is not fixed. In our context, it is more natural to consider the case of a fixed dimension, since in Problems 2 –4 we always fix an underlying group. Using some recent result on the Parikh images of regular languages, we can show:

Theorem 9. *For every finitely generated abelian group the rational subset membership problem can be solved in polynomial time.*

Proof. Consider a fixed finitely generated abelian group $G = \prod_{i=1}^{n} Z_i$, where every Z_i is cyclic. By Theorem 16 from the next section, we can assume that $Z_i \cong \mathbb{Z}$ for every $1 \leq i \leq n$. Take the generating set $X = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}$, where x_i generates Z_i as a group. As usual, let $\pi : X^* \to G$ be the canonical morphism. Recall that the Parikh image of a language $L \subseteq X^*$ is the image of L under the canonical morphism $\Psi : X^* \to \mathbb{N}^{2n}$. Thus, if $\Psi(w) = (c_1, d_1, \ldots, c_n, d_n)$, then c_i (resp., d_i) is the number of occurrences of the symbol x_i (resp., x_i^{-1}) in the word w. It is well-known that the Parikh image $\Psi(L)$ of a regular language (and even a context-free language) is semilinear, i.e., $\Psi(L)$ can be written as

$$\Psi(L) = \bigcup_{i=1}^{k} \{\mathbf{a}_i + \lambda_1 \mathbf{a}_{i,1} + \cdots + \lambda_{l_i} \mathbf{a}_{i,l_i} \mid \lambda_1, \ldots, \lambda_{l_i} \in \mathbb{N}\},$$

for $\mathbf{a}_i, \mathbf{a}_{i,1}, \ldots, \mathbf{a}_{i,l_i} \in \mathbb{N}^{2n}$. It has been recently shown that from a given finite automaton $\mathcal{A}$ over X one can compute a semi-linear representation of the Parikh image $\Psi(L(\mathcal{A})$ in polynomial time [28].[2] Such a semi-linear representation consists of a list of all vectors $\mathbf{a}_i, \mathbf{a}_{i,j}$ ($1 \leq i \leq k$, $1 \leq j \leq l_i$), where the vector entries are represented as binary encoded numbers. It is crucial here that the alphabet Σ is fixed, because the running time of the algorithm from [28] is exponential in the size of the alphabet.

Let us now consider the rational subset membership problem for G. Let $\mathcal{A}$ be a finite automaton over X. We have to check, whether $1 \in \pi(L(\mathcal{A}))$. First, we compute in polynomial time the Parikh image

$$\Psi(L) = \bigcup_{i=1}^{k} \{\mathbf{a}_i + \lambda_1 \mathbf{a}_{i,1} + \cdots + \lambda_{l_i} \mathbf{a}_{i,l_i} \mid \lambda_1, \ldots, \lambda_{l_i} \in \mathbb{N}\}.$$

For a vector $\mathbf{a} = (c_1, d_1, \ldots, c_n, d_n) \in \mathbb{Z}^{2n}$ define the vector $\mathbf{a}' = (c_1 - d_1, \ldots, c_n - d_n) \in \mathbb{Z}^n$. Then, we have $1 \in \pi(L(\mathcal{A}))$ if and only if there are $1 \leq i \leq k$ such that the system

$$\lambda_1 \mathbf{a}'_{i,1} + \cdots + \lambda_{l_i} \mathbf{a}'_{i,l_i} = -\mathbf{a}'_i$$

has a solution in $\mathbb{N}$. But this is an instance of integer programming in the fixed dimension n, which can be solved in polynomial time [30, Sec. 4]. ☐

Let us no come to classical undecidablity results in the context of rational subsets. The first such result was shown by Mihailova [43] in 1966:

Theorem 10. *The direct product $F_2 \times F_2$ of two copies of the free group of rank 2 contains a fixed finitely generated subgroup with an undecidable membership problem.*

In particular, $F_2 \times F_2$ has an undecidable subgroup membership problem. Hence, also the submonoid membership problem and the rational subset membership problem for $F_2 \times F_2$ are undecidable. Mihalova's result is also remarkable since $F_2 \times F_2$ is a very natural group. In contrast all known examples of finitely presented groups with an undecidable word problem are constructed from Turing machines (or other universal computation models) with undecidable acceptance problem and cannot be considered as simple

[2]In particular, all numbers $k, l_1, \ldots, l_k$ are polynomially bounded in the size of the automaton $\mathcal{A}$.

or natural. Nevertheless, Mihailova's result is shown by reducing the word problem for a finitely presented group to the membership problem for a finitely generated subgroup of $F_2 \times F_2$.

A second classical undecidability result for the subgroup membership problem was shown by Rips in 1982:

Theorem 11. *There is a word-hyperbolic group that contains a finitely generated subgroup with an undecidable membership problem.*

So again, the subgroup membership problem, the submonoid membership problem, and the rational subset membership problem are in general undecidable for word-hyperbolic groups. The group constructed by Rips is actually a torsion-free small cancellation group satisfying the condition $C'(1/6)$. Wise modified Rips' construction so that the resulting group is also residually finite [58].

5 Closure properties

For every group theoretic decision problem, let us call it $\mathcal{P}$, it is good to know closure properties with respect to group theoretic constructions. They allow us to construct from groups for which $\mathcal{P}$ is decidable new (and maybe more complicated) groups for which $\mathcal{P}$ is decidable. Mihailova's result (Theorem 10) implies that the class of groups for which the subgroup membership problem (or the submonoid membership problem, or the rational subset membership problem) is decidable is not closed under direct products: F_2 has a decidable rational subset membership problem by Benois' result (Theorem 7) but $F_2 \times F_2$ has an undecidable subgroup membership problem. Another important operation, which destroys the decidability of the rational subset membership problem is the wreath product; see Section 9 for more details. But fortunately, there are other important group constructions for which we can prove positive results.

Two very important constructions in combinatorial group theory are HNN-extensions and amalgamated free products. The following two results were shown in [26] (and independently in [32]) for the rational subset membership problem and in [27] for the subgroup membership problem.

Theorem 12. *Let $\mathcal{P}$ stand for either the rational subset membership problem or the subgroup membership problem. Assume that G is a finitely generated* *group for which $\mathcal{P}$ is decidable. Then $\mathcal{P}$ is decidable for every HNN-extension $\langle G, t \mid t^{-1}at = \varphi(a)\ (a \in A)\rangle$ with $A \leq G$ finite.*

Theorem 13. *Let $\mathcal{P}$ stand for either the rational subset membership problem or the subgroup membership problem. Assume that G_1 and G_2 are finitely generated groups for which $\mathcal{P}$ is decidable. Then $\mathcal{P}$ is decidable for every amalgamated free product $G_1 *_{A_1=A_2} G_2$ with $A_1 \leq G_1$ and $A_2 \leq G_2$ finite.*

Closure of the class of groups with a decidable subgroup membership problem under free products was already shown by Mihailova in [42].

Theorems 12 and 13 can be rephrased in terms of graphs of groups. Every fundamental group of a graph of groups with finite edge groups and vertex groups that have a decidable rational subset membership problem (resp., subgroup membership problem) has a decidable rational subset membership problem (resp., subgroup membership problem) as well.

Surprisingly, it is not known whether the decidability of the submonoid membership problem is preserved under HNN-extensions with finite associated subgroups and amalgamated free products over finite subgroups:

Open problem 14. *Assume that G is a finitely generated group with a decidable submonoid membership problem, and let $H = \langle G, t \mid t^{-1}at = \varphi(a)\ (a \in A)\rangle$ be an HNN-extension with $A \leq G$ finite. Does H have a decidable submonoid membership problem?*

Assume that G_1 and G_2 are finitely generated groups with a decidable submonoid membership problem, and let $G = G_1 *_{A_1=A_2} G_2$ be an amalgamated free product with $A_1 \leq G_1$ and $A_2 \leq G_2$ finite. Does G have a decidable submonoid membership problem? Does the free product $G_1 * G_2$ has a decidable submonoid membership problem?

Actually, the author conjectures that there are specific groups, where the answers to the above questions are negative. We will come back to this conjecture in Section 10 when we consider the relationship between the rational subset membership problem and the submonoid membership problem in more detail.

Let us now discuss subgroups and extensions. The following result is trivial:

Proposition 15. *Let $\mathcal{P}$ stand for either the rational subset membership problem, the submonoid membership problem, or the subgroup membership problem. Assume that H is a finitely generated subgroup of the finitely generated group G. If $\mathcal{P}$ is decidable for G, then $\mathcal{P}$ is decidable for H as well.*

Our last closure result concerns finite extensions and was shown by Grunschlag in his thesis [19]:

Theorem 16. *Let $\mathcal{P}$ stand for either the rational subset membership problem or the subgroup membership problem. Assume that G is a finite index subgroup of H. If $\mathcal{P}$ is decidable for G, then $\mathcal{P}$ is decidable for H as well. Moreover, if $\mathcal{P}$ can be solved in polynomial time for G, then the same holds for the group H.*

Let us sketch the proof. Assume that G (resp., H) is generated by the symmetric set X (resp., Y). Let $W_X(G) \subseteq X^*$ (resp., $W_Y(H) \subseteq Y^*$) be the set of all words that evaluate to the identity of G (resp., H). There exists a rational transduction $\tau \subseteq X^* \times Y^*$ (which is just a rational subset of the monoid $X^* \times Y^*$) such that

$$W_Y(H) = \tau(W_X(G)) = \{w \in Y^* \mid \exists u \in W_X(G) : (u, w) \in \tau\},$$

see [26, Lemma 3.3]. This rational transduction is given by a fixed automaton $\mathcal{T}$ with transitions labelled by pairs from $(X \times \{\varepsilon\}) \cup (\{\varepsilon\} \times Y)$. Here, "fixed" means that we do not have to construct the automaton $\mathcal{T}$.

Take a finite automaton $\mathcal{A}$ over Y. We have to check, whether $L(\mathcal{A})$ contains a word that evaluates to the identity of H, i.e., that belongs to $W_Y(H)$. We have $L(\mathcal{A}) \cap W_Y(H) \neq \emptyset$ if and only if $L(\mathcal{A}) \cap \tau(W_X(G)) \neq \emptyset$ if and only if $\tau^{-1}(L(\mathcal{A})) \cap W_X(G) \neq \emptyset$. Finally, an automaton for $\tau^{-1}(L(\mathcal{A}))$ can be constructed in polynomial time from the automaton $\mathcal{A}$ using a product construction with the automaton $\mathcal{T}$.

As for HNN-extensions and amalgamated free products, it is open whether the decidability of the submonoid membership problem is preserved by finite extensions:

Open problem 17. *Assume that G is a finite index subgroup of H and that G has a decidable submonoid membership problem. Is the submonoid membership problem for H decidable?*

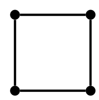
6 Right-angled Artin groups

Let $H = (\Gamma, E)$ be a finite simple graph. In other words, the edge relation $E \subseteq V \times V$ is irreflexive and symmetric. One associates with H the group

$$\mathbb{G}(H) = \langle \Gamma \mid ab = ba \ ((a, b) \in E) \rangle.$$

Such a group is called a *right-angled Artin group*, *graph group*, or *free partially commutative group*. Here, we use the term right-angled Artin group.[3] Right-angled Artin groups received a lot of attention in group theory during the last few years, mainly due to their rich subgroup structure [8, 12, 17].

For graphs $H_1 = (V, E)$ and H_2, we say that H_1 contains an induced H_2, if there is a subset $U \subseteq V$ such that the graph $(U, E \cap (U \times U))$ is isomorphic to H_2. In this situation, $\mathbb{G}(H_2)$ is a subgroup of $\mathbb{G}(H_1)$. With C4 (cycle on 4 nodes) we denote the following graph:



Note that the right-angled Artin group $\mathbb{G}(C4)$ is $F_2 \times F_2$. Hence, by Mihailova's result (Theorem 10), the subgroup membership problem is undecidable for every graph group $\mathbb{G}(H)$ such that H contains an induced C4.

On the decidability side, the following result is shown in [27]. A simplified proof can be found in [34].[4]

Theorem 18. *Let H be a finite simple graph that does not contain an induced cycle on $n \geq 4$ nodes (such a graph is called chordal). Then, the subgroup membership problem for the graph group $\mathbb{G}(H)$ is decidable.*

This result and Mihailova's result leave a gap for the decidability status of the subgroup membership problem.

Open problem 19. *For which graphs H is the subgroup membership problem for the right-angled Artin group $\mathbb{G}(H)$ decidable? More specifically, is the subgroup membership problem decidable for the right-angled Artin group $\mathbb{G}(C5)$ (where C5 denotes a cycle on 5 nodes)?*

With P4 (path on 4 nodes) we denote the following graph:

[3]This term comes from the fact that right-angled Artin groups are exactly the Artin groups corresponding to right-angled Coxeter groups.

[4]Actually, decidability of the subgroup membership problem is shown in [27, 34] for a much larger class of groups.

The following characterization of right-angled Artin groups with a decidable rational subset membership problem (resp., submonoid membership problem) is shown in [33]:

Theorem 20. *Let H be a finite simple graph. Then, the following three conditions are equivalent:*

- *H does not contain an induced $P4$ or $C4$.*
- *The rational subset membership problem for $\mathbb{G}(H)$ is decidable.*
- *The submonoid membership problem for $\mathbb{G}(H)$ is decidable.*

For the undecidability statement in Theorem 20 one has to show that the submonoid membership problem is undecidable for $\mathbb{G}(P4)$ and $\mathbb{G}(C4)$. The latter group is covered by Mihailova's result. For $\mathbb{G}(P4)$ it is first shown in [33] that this group has an undecidable rational subset membership problem. Then, in a second step the rational subset membership problem for $\mathbb{G}(P4)$ is reduced to the submonoid membership problem for $\mathbb{G}(P4)$.[5] To prove that $\mathbb{G}(P4)$ has an undecidable rational subset membership problem, one can use a result from the theory of trace monoid. Trace monoids are the monoid counterparts of right-angled Artin groups. For a finite simple graph $H = (\Gamma, E)$ one defines the corresponding trace monoid $\mathbb{M}(H)$ as the quotient of the free monoid Γ^* by the monoid congruence generated by all pairs (ab, ba) with $(a, b) \in E$. Aalbersberg and Hoogeboom [1] proved that the following two conditions are equivalent:

- It is decidable, whether the intersection of two given rational subsets of the trace monoid $\mathbb{M}(H)$ is nonempty.
- The graph H does not contain an induced P4 or C4.

But for two rational subsets $L, K \subseteq \mathbb{M}(H)$, one has $L \cap K = \emptyset$ if and only if the set LK^{-1} (interpreted in the right-angled Artin group $\mathbb{G}(H)$) contains the identity element 1.

The proof of the decidability statement in Theorem 20 uses the following characterization of graphs without induced P4 or C4, see [59]: A finite simple graph H does not contain an induced P4 or C4 if and only if H can be obtained from the graph with one node using the following two operations:

[5] This reduction is very similar to the reduction of the rational subset membership problem to the submonoid membership problem in case of a group with infinitely many ends. This reduction is outlined in Section 10.

- Take the disjoint union of two graphs.
- Add a new vertex to the graph and connect it to all old nodes.

On the level of right-angled Artin groups, these two operations correspond to (i) the free product of two groups, and (ii) the direct product by $\mathbb{Z}$. Hence, one has to show that the rational subset membership problem is decidable for every group that can be produced from the trivial group 1 using the operations of free product and direct product with $\mathbb{Z}$.

The algorithm from [33] is not very efficient. To deal with the case of a free product, Parikh's theorem (stating that the Parikh image of a context-free language is semi-linear) is applied, which leads to an exponential blow-up in the running time. This implies that for the *uniform* rational subset membership problem for right-angled Artin groups $\mathbb{G}(H)$, where H does not contain an induced P4 or C4 (in this problem, H is also part of the input), the proof in [33] only yields a non-elementary algorithm, i.e., an algorithm whose running time is not bounded by tower of exponents of fixed height.

Open problem 21. *What is the computational complexity of the rational subset membership problem for a right-angled Artin group $\mathbb{G}(H)$, where H does not contain an induced P4 or C4? Is there an algorithm with elementary running time for the uniform problem, where the graph H is part of the input?*

7 Nilpotent groups and polycyclic groups

The lower central series of the group G is the sequence of subgroups $G = G_1 \geq G_2 \geq G_3 \geq \cdots$ where $G_{i+1} = [G_i, G]$ (which is the subgroup of G_i generated by all commutators $g^{-1}h^{-1}gh$ for $g \in G_i$ and $h \in G$; by induction one can show that indeed $G_{i+1} \leq G_i$). The group G is nilpotent if there exists $i \geq 1$ with $G_i = 1$. A group G is polycyclic, if there exists a subnormal series $G = G_0 \rhd G_1 \rhd \cdots \rhd G_{n-1} \rhd G_n = 1$ such that every quotient G_{i-1}/G_i is cyclic. Nilpotent groups are polycyclic.

Mal'cev [39] proved that every polycyclic group G is subgroup separable, i.e., for every finitely generated subgroup $H \leq G$ and $g \in G \setminus H$ there exists a morphism $\varphi : G \to K$ to a finite group K such that $\varphi(g) \notin \varphi(K)$. Together with the finite presentability of finitely generated polycyclic groups, one gets:

Theorem 22. *Every finitely generated polycyclic group has a decidable subgroup membership problem.*

A more practical algorithm for the subgroup membership problem for polycyclic groups can be found in [2].

By Theorem 22 every finitely generated nilpotent group has a decidable subgroup membership problem. This result does not generalize to the rational subset membership problem, as Roman'kov [52] has shown:

Theorem 23. *There exists a number r such that the free nilpotent group of class 2 generated by r elements (this group is denoted by $N_{2,r}$) has an undecidable rational subset membership problem.*

The proof of this result in [52] uses a reduction from Hilbert's 10th problem, i.e., the question whether a Diophantine equation $P(x_1, \ldots, x_n)$ with P a polynomial with integer coefficients has an integer solution. The decidability status of the submonoid membership problem for finitely generated nilpotent groups is open:

Open problem 24. *Is there a finitely generated nilpotent group with an undecidable submonoid membership problem?*

Rational subsets in nilpotent groups were also studied by Bazhenova [4]. She proved that the rational subsets of a finitely generated nilpotent group G are a Boolean algebra if and only if G is virtually abelian.

8 Metabelian groups

Recall that a group G is metabelian if the commutator subgroup $[G, G]$ is abelian. Equivalently, G is metabelian if G has an abelian normal subgroup A such that the quotient G/A is abelian too. Hall [20] has shown that one can view A as a $\mathbb{Z}[Q]$-module, which is finitely generated (as a $\mathbb{Z}[Q]$-module) if G is finitely generated. This fact allows to apply commutative algebra to obtain decidability results for metabelian groups. In particular, in [53, 54] the following result is shown:

Theorem 25. *For every finitely generated metabelian group, the subgroup membership problem is decidable.*

The submonoid membership problem seems to mark the borderline between decidability and undecidability for metabelian groups.

Theorem 26. *The free metabelian group generated by two elements (this group is denoted by M_2 in the following) contains a fixed finitely generated submonoid with an undecidable membership problem.*

This result is shown in [36] via a reduction from the membership problem for finitely generated subsemimodules of free $(\mathbb{Z} \times \mathbb{Z})$-modules of finite rank. This latter problem is shown to be undecidable in [36] by interpreting it as a particular tiling problem of the Euclidean plane[6] that in turn is shown to be undecidable via a direct encoding of a Turing machine.

Also if one tries to generalize Theorem 25 to larger classes of groups, one quickly reaches undecidability, as Umirbaev [57] has shown:

Theorem 27. *The free solvable group of derived length* 3 *and rank* 2 *has an undecidable subgroup membership problem.*

9 Wreath products

Let G and H be groups. Consider the direct sum

$$K = \bigoplus_{g \in G} H_g,$$

where H_g is a copy of H. We view K as the set

$$H^{(G)} = \{f \in H^G \mid f^{-1}(H \setminus \{1\}) \text{ is finite}\}$$

of all mappings from G to H with finite support together with pointwise multiplication as the group operation. The group G has a natural left action on $H^{(G)}$ given by

$$gf(a) = f(g^{-1}a)$$

where $f \in H^{(G)}$ and $g, a \in G$. The corresponding semidirect product $H^{(G)} \rtimes G$ is the wreath product $H \wr G$. In other words:

- Elements of $H \wr G$ are pairs (f, g), where $f \in H^{(G)}$ and $g \in G$.
- The multiplication in $H \wr G$ is defined as follows: Let $(f_1, g_1), (f_2, g_2) \in H \wr G$. Then $(f_1, g_1)(f_2, g_2) = (f, g_1 g_2)$, where $f(a) = f_1(a) f_2(g_1^{-1} a)$.

[6] A good introduction into tiling problems can be found in [9, Appendix A].

The following intuition might be helpful: An element $(f, g) \in H \wr G$ can be thought of as a finite multiset of elements of $H \setminus \{1\}$ that are sitting at certain elements of G (the mapping f) together with the distinguished element $g \in G$, which can be thought of as a cursor moving in G. If we want to compute the product $(f_1, g_1)(f_2, g_2)$, we do this as follows: First, we shift the finite collection of H-elements that corresponds to the mapping f_2 by g_1: If the element $h \in H \setminus \{1\}$ is sitting at $a \in G$ (i.e., $f_2(a) = h$), then we remove h from a and put it to the new location $g_1 a \in H$. This new collection corresponds to the mapping $f_2' \colon a \mapsto f_2(g_1^{-1}a)$. After this shift, we multiply the two collections of H-elements pointwise: If in $a \in G$ the elements h_1 and h_2 are sitting (i.e., $f_1(a) = h_1$ and $f_2'(a) = h_2$), then we put the product $h_1 h_2$ into the location a. Finally, the new distinguished G-element (the new cursor position) becomes $g_1 g_2$.

If H (resp. G) is generated by the set X (resp. Y) with $X \cap Y = \emptyset$, then $H \wr G$ is generated by $X \cup Y$. It is well-known and easy to see that decidability of the word problem for G and H implies decidability of the word problem for $H \wr G$. The following simple proposition is useful, see [37] for a proof:

Proposition 28. *Let K be a subgroup of G of finite index m and let H be a group. Then $H^m \wr K$ is isomorphic to a subgroup of index m in $H \wr G$.*

The following decidability result is shown in [37]:

Theorem 29. *The rational subset membership problem is decidable for every group $H \wr V$ with H finite and V virtually free.*

Note that Theorem 29 covers the well known lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

The proof of Theorem 29 in [37] makes use of well-quasi-order (wqo) theory. Let us briefly explain the idea for a wreath product $G = H \wr F_2$, where H is finite and F_2 is the free group generated by a and b. Given a finite automaton $\mathcal{A}$ over the alphabet $H \cup \{a, a^{-1}, b, b^{-1}\}$ it suffices to check whether $\mathcal{A}$ accepts a word that represents the identity of G. The key ingredient is a certain language over the alphabet of triples (p, d, q), where p and q are states of the automaton $\mathcal{A}$ and $d \in \{a, a^{-1}, b, b^{-1}\}$. The idea is that such a triple may represent a path in $\mathcal{A}$ from state p to q such that the sequence of labels from $\{a, a^{-1}, b, b^{-1}\}$ along the path is a loop in the Cayley-graph of F_2 that leaves the origin in direction d and returns to the origin from direction d. The effect of such a path is the product of all transition labels; it is an element of the direct sum $K = \bigoplus_{g \in F_2} H$. A word w over the alphabet of triples is a *loop pattern* if each triple (p, d, q) in the word can be replaced by an automaton path as described above, such that the product of the effects of these paths is the identity of K. It is shown in [37] that the set of all loop patterns is a regular language. For this, it is shown that the set of loop patterns is an upward closed set of words with respect to a wqo, which is a refinement of the subsequence relation (also known as embeddability) on words (which is a wqo by Higman's Lemma). Using a saturation process one can actually compute an automaton for the set of all loop patterns. Using this automaton, it is straightforward to check whether $\mathcal{A}$ accepts a word that represents the identity of G.

The computational complexity of the rational subset membership problem for groups $H \wr V$ with H finite and V virtually free is open. Due to the use of well quasi orders, the algorithm from [37] is not primitive recursive.

Open problem 30. *Is the rational subset membership problem for groups $H \wr V$ with H finite and V virtually free primitive recursive? In particular, is the rational subset membership problem for the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ primitive recursive?*

It should be mentioned that there exist several decision problems, for which decidability is proved using a well quasi order, and which can be shown to be not primitive recursive. An example is the membership problem for so called leftist grammars (these are grammars, where every production has the form $ab \to b$ or $d \to cd$) [23, 45].

By the following result from [37], decidability for the rational subset membership problem cannot be pushed very far beyond wreath products of the form $H \wr V$ with H finite and V virtually free:

Theorem 31. *There is a fixed finitely generated submonoid M of the wreath product $\mathbb{Z} \wr \mathbb{Z}$ with an undecidable membership problem.*

For the proof of Theorem 31 in [37], the authors encode the acceptance problem for a 2-counter machine (Minsky machine [44]) into the submonoid membership problem for $\mathbb{Z} \wr \mathbb{Z}$. One should remark that $\mathbb{Z} \wr \mathbb{Z}$ is a finitely generated metabelian group and hence has a decidable subgroup membership problem, see Theorem 25.

The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is a subgroup of Thompson's group F (see [41]) as well as of Baumslag's finitely presented metabelian group $\langle a, s, t \mid [s, t] = [a^t, a] = 1, a^s = aa^t \rangle$ [3], see e.g. [11]. Hence, we get:

Corollary 32. *Thompson's group F as well as Baumslag's finitely presented metabelian group both contain finitely generated submonoids with an undecidable membership problem.*

A further undecidability result for wreath products was shown in [36]:

Theorem 33. *For every non-trivial group H, the rational subset membership problem for $H \wr (\mathbb{Z} \times \mathbb{Z})$ is undecidable.*

The proof of this result in [36] uses an encoding of a tiling problem, which uses the grid structure of the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$. It is very similar to the undecidability proof for the submonoid membership problem for free metabelian groups (Theorem 26) It is open, whether Theorem 33 can be sharpened to the submonoid membership problem:

Open problem 34. *Assume that H is a non-trivial group. Is the submonoid membership problem for $H \wr (\mathbb{Z} \times \mathbb{Z})$ undecidable?*

The author conjectures that the answer to this question is positive. Another reasonable conjecture is that Theorem 33 can be generalized to every wreath product $H \wr G$, where H is non-trivial and G is not virtually free (note that $\mathbb{Z} \times \mathbb{Z}$ is not virtually free).

Open problem 35. *Assume that H is a non-trivial group and G is not virtually free. Is the rational subset membership problem for $H \wr G$ undecidable?*

As remarked above, the author conjectures that the answer to this question is again positive. The reason is that the undecidability proof for $H \wr (\mathbb{Z} \times \mathbb{Z})$ from [36] only uses the grid-like structure of the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$. In [29] it was shown that the Cayley graph of a group G has bounded tree width (a graph-theoretic measure that, roughly speaking, determines how tree-like a graph is) if and only if the group is virtually free. Hence, if G is not virtually free, then the Cayley-graph of G has unbounded tree width. By known results from graph theory, this implies that finite grids of arbitrary size appear as graph-theoretic minors in the Cayley-graph of G. There is hope to use these grids for encoding an undecidable tiling problem into the rational subset membership problem for $H \wr G$ (for H non-trivial).

Theorem 31 and 33 imply the following: For finitely generated non-trivial abelian groups G and H, the wreath product $H \wr G$ has a decidable rational subset membership problem if and only if (i) G is finite[7] or (ii) G has rank 1 and H is finite. Furthermore, for virtually free groups G and H, the rational subset membership problem is decidable for $H \wr G$ if and only if (i) G is trivial or (ii) H is finite, or (iii) G is finite and H is virtually $\mathbb{Z}$, i.e., has $\mathbb{Z}$ as a finite index subgroup. Note that if G is finite, then Proposition 28 implies that $H^{|G|}$ is a finite index subgroup of $H \wr G$. Hence, if H is virtually $\mathbb{Z}$, then $H^{|G|}$ is virtually abelian and hence has a decidable rational subset membership problem. On the other hand, if H is virtually F_n for F_n a free group of rank $n > 1$ and G is nontrivial, then $H^{|G|}$ (and hence $H \wr G$) has an undecidable subgroup membership problem by Theorem 10.

10 Rational subsets versus submonoids

It is a trivial obersvation that decidability of the rational subset membership problem for a group G implies decidability of the submonoid membership problem for G, and the latter implies decidability of the subgroup membership problem for G. On the other hand, we have seen groups, for which the subgroup membership problem is decidable, but the submonoid membership problem is undecidable. Examples are the free metabelian group generated by two elements (see Theorems 25 and 26) and the right-angled Artin group $\mathbb{G}(P4)$ (see Theorems 18 and 20; note that $P4$ does not contain an induced cycle, which allows to apply Theorem 18). It is therefore an interesting question, whether there is a finitely generated group, for which the submonoid membership problem is decidable but the rational subset membership problem is undecidable. Unfortunately, we do not know, whether such a group exists.

Open problem 36. *Is there a finitely generated group, for which the submonoid membership problem is decidable but the rational subset membership problem is undecidable?*

By the following result from [35] we know that if such a group exists, then it must have only one end. The number of ends of a finitely generated infinite group G is a geometric invariant of G that is defined as follows: Assume that G is finitely generated by the symmetric set X (the following definition is

[7]If G has size m, then by Proposition 28, $H^m \cong H^m \wr 1$ is isomorphic to a subgroup of index m in $H \wr G$. Since H^m is finitely generated abelian, decidability of the rational subset membership problem of $H \wr G$ follows from Theorems 8 and 16.

not influenced by the concrete choice of X) and consider the Cayley graph $\mathcal{G}(G, X)$. The nodes of this graph are the elements of G and there is an edge between two elements of $g, h \in G$ if and only if there is a generator $a \in X$ such that $h = ga$ in G. This graph is undirected (since X is symmetric) and connected (since X generates G). Moreover, it is vertex-transitive, which means that for all $g, h \in G$, there is a graph automorphism of $\mathcal{G}(G, X)$ that maps g to h. To define the number of ends of G, choose an arbitrary node $g \in G$ (the concrete choice of g is not important) and let $\mathcal{G}_n$ (for $n \geq 0$) be the subgraph of $\mathcal{G}(G, X)$ obtained by removing all nodes from $\mathcal{G}(G, X)$ that have distance at most n from g. Let e_n be the number of connected components of $\mathcal{G}_n$. Then the number of ends is the limit of the sequence $(e_n)_{n \geq 0}$ or ∞ if this sequence is unbounded. By the Freudenthal-Hopf Theorem, every finitely generated infinite group G has either 1, 2, or ∞ many ends, see e.g. [41]. Here are three typical examples for each possibility:

- The number of ends of $\mathbb{Z} \times \mathbb{Z}$ is 1.
- The number of ends of $\mathbb{Z}$ is 2.
- The number of ends of the free group F_2 or rank 2 is ∞.

A group has two ends if and only if it is virtually $\mathbb{Z}$. A seminal result of Stallings [55, 56] characterizes groups with infinitely many ends: A group has infinitely many ends if and only if it is an HNN-extension with finite associated subgroups or an amalgamated product with finite amalgamated subgroups. The following result was shown in [35]:

Theorem 37. *Assume that G is a finitely generated group G. If G has more than one end, then the rational subset membership problem for G is decidable if and only if the submonoid membership problem for G is decidable.*

The case of group G with two ends is easy: G has $\mathbb{Z}$ as a finite index subgroup. Since the rational subset membership problem for $\mathbb{Z}$ is decidable, Theorem 16 implies that the rational subset membership problem (and hence also the submonoid membership problem) is decidable. So, it remains to consider a group G with infinitely many ends. By Stalling's theorem one can write G as an HNN-extension with finite associated subgroups or an amalgamated product with finite amalgamated subgroups. Let us sketch the proof of Theorem 37 in a simple case that nevertheless shows the main idea: Assume that $G = H * F_2$ and assume that the submonoid membership problem for G is decidable. We have to show that G has a decidable rational subset membership problem. By Theorem 13 (and the fact that F_2 has a decidable rational subset membership problem) it suffices to show that H has a decidable rational subset membership problem. So, let us fix a generating set X for H together with a canonical homomorphism $\pi : X^* \to H$, and let $\mathcal{A} = (Q, \Delta, q_0, F)$ be a finite automaton over X. By adding ε-transitions to Δ we can assume that F consists of a single state $q_f \neq q_0$. Since F_2 contains a copy of F_n (the free group of rank n) for any $n \geq 1$, we can assume that F_2 contains a copy of $F(Q)$, i.e., the free group generated by the states of $\mathcal{A}$. Recall that $\Delta \subseteq Q \times (X \cup \{\varepsilon\}) \times Q$ is the set of transitions. Now define a finitely generated submonoid of $G = H * F_2$ as follows. Let

$$Y = \{q^{-1}ap \mid (q, a, p) \in \Delta\} \subseteq H * F(Q) \subseteq H * F_2 = G.$$

Then, one can show that for every $w \in X^*$, we have $\pi(w) \in \pi(L(\mathcal{A}))$ if and only if $q_0^{-1}wq_f$ represents an element of the submonoid Y^* of G. The idea is that in a product of the form $(q_1^{-1}a_1p_1)(q_2^{-1}a_2p_2)\cdots(q_n^{-1}a_np_n)$ a factor of the form $p_iq_{i+1}^{-1}$ with $p_i \neq q_{i+1}$ cannot be erased. On the other hand, if $p_i = q_{i+1}$ for $1 \leq i \leq n-1$, then the word is equal to $q_1^{-1}(a_1a_2\cdots a_n)p_n$.

Problem 36 is related to Problem 14: Assume that the class of finitely generated groups with a decidable submonoid membership problem is closed under free product (whether this is true was one of the questions asked in Problem 14). Let G be an arbitrary finitely generated group with a decidable submonoid membership problem. Hence, by our assumption, also the free product $G * F_2$ has a decidable submonoid membership problem. But this group has infinitely many ends. So, by Theorem 37, $G * F_2$ has a decidable rational subset membership problem. But then, also the finitely generated subgroup G has a decidable rational subset membership problem.

The author conjectures that one can construct a finitely generated group with a decidable submonoid membership problem and an undecidable rational subset membership problem. This leads to the conjecture that the class of groups with a decidable submonoid membership problem is not closed under free products.

11 Further results on the submonoid membership problem

Let us briefly mention some further results on the submonoid membership problem. In [46] the bounded submonoid membership problem for a finitely generated group G was introduced:

Decision problem 38. Bounded submonoid membership problem for G

- *INPUT: Elements $g, g_1, \ldots, g_n \in G$ and a unary encoded number k*
- *QUESTION: Can g be written as a product $g = g_{i_1} g_{i_2} \cdots g_{i_l}$ with $l \leq k$ and $1 \leq i_1, \ldots, i_l \leq n$.*

It was shown in [46] that the bounded submonoid membership problem can be solved in polynomial time for finitely generated virtually nilpotent groups and word hyperbolic groups.

In [22] it was shown that the word problem for a one-relator inverse monoid $\mathsf{Inv}\langle X \mid r = 1\rangle$ is decidable if and only if the submonoid of the one-relator group $\mathsf{Gp}\langle X \mid r = 1\rangle$ that is generated by all prefixes of r has a decidable membership problem. The latter problem is also called the *prefix monoid membership problem* for the one-relator group $\mathsf{Gp}\langle X \mid r = 1\rangle$. Motivated by this result, the submonoid membership problem was further studied in [40], where a general technique based on distortion functions for solving submonoid membership problems is introduced. Using this technique, the authors show that the prefix membership problem is decidable for Baumslag-Solitar groups, surface groups of genius at least two (for which decidability was already shown in [22]), and certain one-relator groups given by Adian type presentations.

12 The rational subset membership problem for monoids and semigroups

We defined the notion of a rational subset for all monoids. Hence, it makes sense to study the rational subset membership problem for finitely generated monoids (and, by replacing the monoid closure by the semigroup closure, even for finitely generated semigroups). Kambites and Render proved several interesting results in this context. They showed that the rational subset membership problem is decidable for the following classes of finitely generated monoids:

- Polycyclic and bicyclic monoids [49],
- Finitely generated Rees matrix semigroups (with or without zero) over a semigroup with decidable rational subset membership problem [50],
- Monoid that satisfy the small overlap condition $C(4)$ (which is inspired by small cancellation theory for groups) [25].

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