Satisfiability of ECTL with tree constraints*

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Abstract. Recently, we proved that satisfiability for ECTL^* with constraints over $\mathbb{Z}$ is decidable using a new technique based on weak monadic second-order logic with the bounding quantifier (WMSO+B). Here we apply this approach to concrete domains that are tree-like. We show that satisfiability of ECTL^* with constraints is decidable over (i) semi-linear orders, (ii) ordinal trees (semi-linear orders where the branches form ordinals), and (iii) infinitely branching order trees of height h for each fixed $h \in \mathbb{N}$. In contrast, we introduce Ehrenfeucht-Fraïssé-games for WMSO+B (weak MSO with the bounding quantifier) and use them to show that our approach cannot deal with the class of order trees. **Missing proofs and details can be found in the long version [6].**

1 Introduction

Temporal logics like LTL, CTL or CTL^* are nowadays standard languages for specifying system properties in verification. These logics are interpreted over node labeled graphs (Kripke structures), where the node labels represent abstract properties of a system. Clearly, such an abstracted system state does in general not contain all the information of the original system state. This may lead to incorrect results in model-checking. To overcome this problem, extensions of temporal logics with constraints have been studied. In this setting, a model of a formula is not only a Kripke structure but a Kripke structure where every node is assigned several values from some fixed structure $\mathcal{C}$ (called a concrete domain). The logic is then enriched in such a way that it has access to the relations of the concrete domain. For instance, if $\mathcal{C} = (\mathbb{Z}, =)$ then every node of the Kripke structure gets assigned several integers and the logic can compare the integers assigned to neighboring nodes for equality.

In our recent papers [4,5] we used a new method (called EHD-method in the following) to show decidability of the satisfiability problem for extended computation tree logic (ECTL^*, which strictly extends CTL^*) with local constraints over the integers. This result greatly improves the partial results on fragments of CTL^* obtained in [2,3,8]. The idea of the EHD-method is as follows. Let $\mathcal{C}$ be any concrete domain over a relational signature σ. Then, satisfiability of ECTL^* with constraints over $\mathcal{C}$ is decidable if $\mathcal{C}$ has the following two properties:

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- The structure $\mathcal{C}$ is negation-closed, i.e., the complement of any relation $R \in \sigma$ is definable in positive existential first-order logic.
- There is a $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO}{+}\mathsf{B})$-sentence φ such that for any countable σ-structure $\mathcal{A}$ there is a homomorphism from $\mathcal{A}$ to $\mathcal{C}$ if and only if $\mathcal{A} \models \varphi$.

Here, $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO}{+}\mathsf{B})$ is the set of all Boolean combinations of MSO-formulas and formulas of WMSO+B, i.e., weak monadic second-order logic with the bounding quantifier. The latter allows to express that there is a bound on the size of finite sets satisfying a certain property. Our decidability result uses the main result of [1] stating that satisfiability of WMSO+B over infinite trees is decidable. In [4] we proved that the existence of a homomorphism into $(\mathbb{Z}, <, =)$ can be expressed in $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO}{+}\mathsf{B})$, showing that ECTL^* with constraints over this structure is decidable.

These results gave rise to the hope that the EHD-method applies to other concrete domains. An interesting candidate in this setting is the infinite order tree $\mathcal{T}_\infty = (\mathbb{N}^*, <, \perp, =)$, where $<$ denotes the prefix order on $\mathbb{N}^*$ and $\perp$ denotes the incomparability relation with respect to $<$ (this structure is negation-closed, which is the reason for adding the incomparability relation $\perp$). Unfortunately, this hope is destroyed by one of the main results of this work, which is shown in Section 5 using a new Ehrenfeucht-Fraïssé-game for WMSO+B:

Theorem 1. *There is no* $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO}{+}\mathsf{B})$*-sentence ψ such that for every countable structure $\mathcal{A}$ (over the signature $\{<, \perp, =\}$) we have: $\mathcal{A} \models \psi$ if and only if there is a homomorphism from $\mathcal{A}$ to $\mathcal{T}_\infty$.*

Thm. 1 shows that the EHD-method cannot be applied to the concrete domain $\mathcal{T}_\infty$. Of course, this does not imply that satisfiability for ECTL^* with constraints over $\mathcal{T}_\infty$ is undecidable, which remains an open problem (even for LTL instead of ECTL^*). In fact, we conjecture that satisfiability for ECTL^* with constraints over $\mathcal{T}_\infty$ is decidable. We support this conjecture by applying the EHD-method to other tree-like structures, such as semi-linear orders, ordinal trees, and infinitely branching trees of a fixed height. *Semi-linear orders* are partial orders that are tree-like in the sense that for every element x the set of all smaller elements form a linear suborder. If this linear suborder is an ordinal (for every x) then one has an *ordinal tree*. Ordinal trees are widely studied in descriptive set theory and recursion theory. Note that a tree is a connected semi-linear order where for every element the set of all smaller elements is finite.

In the integer-setting from [4,5], we investigated satisfiability for ECTL^*-formulas with constraints over one fixed structure (integers with additional relations). For semi-linear orders and ordinal trees it is more natural to consider satisfiability with respect to a class of concrete domains Γ (over a fixed signature σ): The question becomes, whether for a given constraint ECTL^* formula φ there is a concrete domain $\mathcal{C} \in \Gamma$ such that φ is satisfiable by some model with concrete values from $\mathcal{C}$? If a class Γ has a universal structure[3] $\mathcal{U}$, then satisfiability with

[3] A structure $\mathcal{U}$ is universal for a class Γ if there is a homomorphic embedding of every structure from Γ into $\mathcal{U}$ and $\mathcal{U}$ belongs to Γ.

respect to the class Γ is equivalent to satisfiability with respect to $\mathcal{U}$ because obviously a formula φ has a model with some concrete domain from Γ if and only if it has a model with concrete domain $\mathcal{U}$. A typical class with a universal model is the class of all countable linear orders, for which $(\mathbb{Q}, <)$ is universal. Similarly, for the class of all countable trees the tree $\mathcal{T}_\infty$ as well as the binary infinite tree are universal. There is also a universal countable semi-linear order. We formulate our decidability result for classes instead of universal structures because there is no universal structure for the class of countable ordinal trees (for a similar reason as the one showing that the class of countable ordinals does not contain a universal structure).

Application of the EHD-method to semi-linear orders and ordinal trees gives the following decidability results.

Theorem 2. *Satisfiability of* ECTL^**-formulas with constraints over each of the following classes is decidable:*

(1) the class of all semi-linear orders (see Section 3),
(2) the class of all ordinal trees (see Section 4), and
(3) for each $h \in \mathbb{N}$*, the class of all order trees of height* h *(see Section 4).*

Concerning complexity, let us remark that in [4,5] we did not present an upper bound on the complexity of our decision procedure. The reason for this is that there is no known upper bound for the complexity of satisfiability of $\mathsf{WMSO}+\mathsf{B}$ over infinite trees, even in the case that the input formula has bounded quantifier depth. Here, the situation is different. Our applications of the EHD-method for Thm. 2 do not use the bounding quantifier whence classical WMSO (for (1)) and MSO (for (2) and (3)) suffice. Moreover, the formulas that express the existence of a homomorphism have only small quantifier depth (at least for semi-linear orders and ordinal trees; for trees of bounded height, the quantifier depth depends on the height). These facts yield a triply exponential upper bound on the time complexity in (1) and (2) from Thm. 2. We skip the proof details, since we still conjecture the exact complexity to be doubly exponential.

2 Preliminaries

In this section we recall basics concerning Kripke structures, various classes of tree-like structures, and the logics MSO, $\mathsf{WMSO}+\mathsf{B}$, and ECTL^* with constraints.

2.1 Structures

Let P be a countable set of atomic propositions. A Kripke structure over P is a triple $\mathcal{K} = (D, \rightarrow, \rho)$, where: (i) D is an arbitrary set of nodes, (ii) $\rightarrow$ is a binary relation on D such that for all $u \in D$ there is a $v \in D$ with $u \rightarrow v$, and (iii) $\rho : D \rightarrow 2^{\mathsf{P}}$ is a function that assigns to every node a set of atomic propositions.

A (finite relational) signature is a finite set $\sigma = \{r_1, \ldots, r_n\}$ of relation symbols. Every relation symbol $r \in \sigma$ has an associated arity $\mathsf{ar}(r) \geq 1$. A σ-structure is a pair $\mathcal{A} = (A, I)$, where A is a non-empty set and I maps every $r \in \sigma$

to an $\mathsf{ar}(r)$-ary relation over A. Quite often, we will identify the relation $I(r)$ with the relation symbol r, and we will specify a σ-structure as $(A, r_1, \dots, r_n)$. Given $\mathcal{A} = (A, r_1, \dots, r_n)$ and given a subset B of A, we define $r_{i\restriction B} = r_i \cap B^{\mathsf{ar}(r_i)}$ and $\mathcal{A}_{\restriction B} = (B, r_{1\restriction B}, \dots, r_{n\restriction B})$ (the *restriction of $\mathcal{A}$ to the set B*). For a subsignature $\tau \subseteq \sigma$, a τ-structure $\mathcal{B} = (B, J)$ and a σ-structure $\mathcal{A} = (A, I)$, a *homomorphism* from $\mathcal{B}$ to $\mathcal{A}$ is a mapping $h : B \to A$ such that for all $r \in \tau$ and all tuples $(b_1, \dots, b_{\mathsf{ar}(r)}) \in J(r)$ we have $(h(b_1), \dots, h(b_{\mathsf{ar}(r)})) \in I(r)$. We write $\mathcal{B} \preceq \mathcal{A}$ if there is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$. Note that we do not require this homomorphism to be injective.

We now introduce *constraint graphs*. These are two-sorted structures where one part is a Kripke structure and the other part is some σ-structure called the *concrete domain*. To connect the concrete domain with the Kripke structure, we fix a set of unary function symbols $\mathcal{F}$. The interpretation of a function symbol from $\mathcal{F}$ is a mapping from the nodes of the Kripke structure to the universe of the concrete domain. Constraint graphs are the structures in which we evaluate constraint ECTL^*-formulas. Formally, an *$\mathcal{A}$-constraint graph* $\mathbb{C}$ is a tuple $(\mathcal{A}, \mathcal{K}, (f^{\mathbb{C}})_{f\in\mathcal{F}})$ where: (i) $\mathcal{A} = (A, I)$ is a σ-structure (the concrete domain), (ii) $\mathcal{K} = (D, \to, \rho)$ is a Kripke structure, and (iii) for each $f \in \mathcal{F}$, $f^{\mathbb{C}} : D \to A$ is the interpretation of the function symbol f connecting elements of the Kripke structure with elements of the concrete domain. An $\mathcal{A}$-constraint path $\mathbb{P}$ is an $\mathcal{A}$-constraint graph of the form $\mathbb{P} = (\mathcal{A}, \mathcal{P}, (f^{\mathbb{P}})_{f\in\mathcal{F}})$, where $\mathcal{P} = (\mathbb{N}, \mathsf{S}, \rho)$ is a Kripke structure such that S is the successor relation on $\mathbb{N}$.

We use $(\mathcal{A}, \mathcal{K}, \mathcal{F}^{\mathbb{C}})$ as an abbreviation for $(\mathcal{A}, \mathcal{K}, (f^{\mathbb{C}})_{f\in\mathcal{F}})$. Moreover, we often drop the superscript $\mathbb{C}$ and also write constraint graph instead of $\mathcal{A}$-constraint graph if no confusion arises.

2.2 Tree-like structures

A *semi-linear order* is a partial order $\mathcal{P} = (P, <)$ with the additional property that for all $p \in P$ the suborder induced by $\{p' \in P \mid p' \leq p\}$ forms a linear order. Note that all (order) trees are semi-linear orders, but not vice-versa. We call a semi-linear order $\mathcal{P} = (P, <)$ an *ordinal forest* (resp., *forest*) if for all $p \in P$ the suborder induced by $\{p' \in P \mid p' \leq p\}$ is an ordinal (resp., a finite linear order). A (ordinal) forest is a *(ordinal) tree* if it has a unique minimal element. A tree has *height* h (for $h \in \mathbb{N}$) if it contains a linear suborder with $h+1$ many elements but no linear suborder with $h+2$ elements.

Given a partial order $(P, <)$, we denote by $\perp_<$ the *incomparability relation* defined by $p \perp_< q$ iff neither $p \leq q$ nor $q \leq p$. Given a $\{<, \perp, =\}$-structure $\mathcal{P} = (P, <, \perp, =)$ such that $(P, <)$ is a semi-linear order (resp., ordinal tree, tree of height h), $=$ is the equality relation on P, and $\perp = \perp_<$, then we also say that $\mathcal{P}$ is a semi-linear order (resp. ordinal tree, tree of height h).

2.3 Logics

As usual, MSO denotes *monadic second-order logic* and WMSO its variant *Weak monadic second-order logic* where set quantifiers only range over finite sets.

Throughout the paper Var_1 (Var_2) denotes the set of element (set, resp.) variables. Finally, $\mathsf{WMSO{+}B}$ is the extension of WMSO by the *bounding quantifier* $\mathsf{B}X\,\varphi$ (see [1]) whose semantics is given by $(A, I) \models \mathsf{B}X\,\varphi(X)$ if and only if there is a bound $b \in \mathbb{N}$ such that $|B| \leq b$ for every finite subset $B \subseteq A$ with $(A, I) \models \varphi(B)$. The *quantifier rank* of a $\mathsf{WMSO{+}B}$-formula is the maximal number of nested quantifiers (existential, universal, and bounding quantifiers) in the formula. We write $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO{+}B})$ for the set of all boolean combinations of MSO-formulas and $\mathsf{WMSO{+}B}$-formulas.

Extended computation tree logic (ECTL^*) is an extension of CTL^* introduced in [9,10]. Like CTL^*, ECTL^* is interpreted on Kripke structures, but while CTL^* allows to specify LTL properties of infinite paths of such models, ECTL^* can describe regular (i.e., MSO-definable) properties of paths. In [5] we introduced an extension of ECTL^*, called constraint ECTL^*, which enriches ECTL^* by local constraints in path formulas.

We now first recall the definition of constraint path MSO-formulas, which take the role of path formulas in constraint ECTL^*. Since we exclusively consider tree-like concrete domains over the fixed signature $\tau = \{<, \perp, =\}$ we only introduce *Constraint path* MSO *(over a signature* τ*)*, denoted as $\mathsf{MSO}(\tau)$.[4] This is the usual MSO for (colored) infinite paths (also known as word structures) with a successor function S extended by atomic formulas that describe local constraints over the concrete domain. Thus, $\mathsf{MSO}(\tau)$ is evaluated over the class of $\mathcal{A}$-constraint paths for any τ-structure $\mathcal{A}$. So fix a set P of atomic propositions and a set $\mathcal{F}$ of unary function symbols. Formulas of $\mathsf{MSO}(\tau)$ are defined by the following grammar:

$$\psi \ ::= \ p(x) \mid S^i(x) = S^j(y) \mid x \in X \mid \neg\psi \mid (\psi \wedge \psi) \mid \exists x\, \psi \mid \exists X\, \psi \mid f_1\mathsf{S}^i(x) \circ f_2\mathsf{S}^j(x)$$

where $\circ \in \tau$, $p \in \mathsf{P}$, $x, y \in \mathsf{Var}_1$, $X \in \mathsf{Var}_2$, $i, j \in \mathbb{N}$ and $f_1, f_2 \in \mathcal{F}$. We call formulas of the form $f_1\mathsf{S}^i(x) \circ f_2\mathsf{S}^j(x)$ for $\circ \in \tau$ *atomic constraints*. It is important to notice that in an atomic constraint only one first-order variable x is used.

Let $\mathbb{P} = (\mathcal{A}, \mathcal{P}, (f^{\mathbb{P}})_{f\in\mathcal{F}})$ be an $\mathcal{A}$-constraint path where $\mathcal{P} = (\mathbb{N}, \mathsf{S}, \rho)$, and let $\eta : (\mathsf{Var}_1 \cup \mathsf{Var}_2) \to (\mathbb{N} \cup 2^{\mathbb{N}})$ be a valuation function mapping first-order variables to elements and second-order variables to sets. The satisfaction relation $\models$ is defined by induction as follows (we omitted the obvious cases for $\neg$ and $\wedge$):

- $(\mathbb{P}, \eta) \models p(x)$ iff $p \in \rho(\eta(x))$.
- $(\mathbb{P}, \eta) \models \mathsf{S}^i(x_1) = \mathsf{S}^j(x_2)$ iff $\eta(x_1) + i = \eta(x_2) + j$.
- $(\mathbb{P}, \eta) \models x \in X$ iff $\eta(x) + i \in \eta(X)$.
- $(\mathbb{P}, \eta) \models \exists x\, \psi$ iff there is an $n \in \mathbb{N}$ such that $(\mathbb{P}, \eta[x \mapsto n]) \models \psi$.
- $(\mathbb{P}, \eta) \models \exists X\, \psi$ iff there is an $E \subseteq \mathbb{N}$ such that $(\mathbb{P}, \eta[X \mapsto E]) \models \psi$.
- $(\mathbb{P}, \eta) \models f_1\mathsf{S}^i(x) \circ f_2\mathsf{S}^j(x)$ iff $\mathcal{A} \models f_1^{\mathbb{P}}(\eta(x) + i) \circ f_2^{\mathbb{P}}(\eta(x) + j)$.

For an $\mathsf{MSO}(\tau)$-formula ψ the satisfaction relation only depends on the free variables of ψ. This motivates the following notation: If $\psi(X_1, \ldots, X_m)$ is an $\mathsf{MSO}(\tau)$-formula where $X_1, \ldots, X_m \in \mathsf{Var}_2$ are the only free variables, we write

[4] For a presentation of the general case we refer the reader to [5]

$\mathbb{P} \models \psi(A_1, \ldots, A_m)$ if and only if, for every valuation function η such that $\eta(X_i) = A_i$, we have $(\mathbb{P}, \eta) \models \psi$.

Having defined $\mathsf{MSO}(\tau)$-formulas we are ready to define constraint ECTL^* over the signature τ (denoted by $\mathsf{ECTL}^*(\tau)$):

$$\varphi ::= \mathsf{E}\psi(\underbrace{\varphi, \ldots, \varphi}_{m \text{ times}}) \mid (\varphi \wedge \varphi) \mid \neg\varphi$$

where $\psi(X_1, \ldots, X_m)$ is an $\mathsf{MSO}(\tau)$-formula in which at most the second-order variables $X_1, \ldots, X_m \in \mathsf{Var}_2$ are allowed to occur freely.

$\mathsf{ECTL}^*(\tau)$-formulas are evaluated over nodes of $\mathcal{A}$-constraint graphs. Let $\mathbb{C} = (\mathcal{A}, \mathcal{K}, (f^{\mathbb{C}})_{f \in \mathcal{F}})$ be an $\mathcal{A}$-constraint graph, where $\mathcal{K} = (D, \rightarrow, \rho)$. We define an infinite path π in $\mathcal{K}$ as a mapping $\pi : \mathbb{N} \to D$ such that $\pi(i) \to \pi(i+1)$ for all $i \geq 0$. For an infinite path π in $\mathcal{K}$ we define the infinite constraint path $\mathbb{P}_\pi = (\mathcal{A}, (\mathbb{N}, \mathsf{S}, \rho'), (f^{\mathbb{P}_\pi})_{f \in \mathcal{F}})$, where $\rho'(n) = \rho(\pi(n))$ and $f^{\mathbb{P}_\pi}(n) = f^{\mathbb{C}}(\pi(n))$. Note that we may have $\pi(i) = \pi(j)$ for $i \neq j$. Given $d \in D$ and an $\mathsf{ECTL}^*(\tau)$-formula φ, we define $(\mathbb{C}, d) \models \varphi$ inductively (again omitting the obvious cases for $\neg$ and $\wedge$) by $(\mathbb{C}, d) \models \mathsf{E}\psi(\varphi_1, \ldots, \varphi_m)$ iff there is an infinite path π in $\mathcal{K}$ with $d = \pi(0)$ and $\mathbb{P}_\pi \models \psi(A_1, \ldots, A_m)$, where $A_i = \{j \mid j \geq 0, (\mathbb{C}, \pi(j)) \models \varphi_i\}$. Note that for checking $(\mathbb{C}, d) \models \varphi$ we may ignore all propositions $p \in \mathsf{P}$ and all functions $f \in \mathcal{F}$ that do not occur in φ.

Given a class of τ-structures Γ, $\mathsf{SAT\text{-}ECTL}^*(\Gamma)$ denotes the following computational problem: *Given a formula* $\varphi \in \mathsf{ECTL}^*(\tau)$*, is there a concrete domain* $\mathcal{A} \in \Gamma$ *and a constraint graph* $\mathbb{C} = (\mathcal{A}, \mathcal{K}, (f^{\mathbb{C}})_{f \in \mathcal{F}})$ *such that* $\mathbb{C} \models \varphi$*?* We also write $\mathsf{SAT\text{-}ECTL}^*(\mathcal{A})$ instead of $\mathsf{SAT\text{-}ECTL}^*(\{\mathcal{A}\})$.

2.4 Constraint ECTL^* and definable homomorphisms

Remember that we focus our interest in this paper on the satisfiability problem with respect to a class of structures over the signature $\tau = \{<, \perp, =\}$ where $=$ is always interpreted as equality and $\perp$ as the incomparability relation with respect to $<$. In [5], we provided a connection between $\mathsf{SAT\text{-}ECTL}^*(\mathcal{A})$ for a τ-structure $\mathcal{A}$ and the definability of homomorphisms to $\mathcal{A}$ in the logic $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO{+}B})$. To be more precise, we are interested in definability of homomorphisms to the $\{<, \perp\}$-reduct of $\mathcal{A}$. In order to facilitate the presentation of this connection, we fix a class Γ of $\{<, \perp\}$-structures.

For every $\mathcal{A} = (A, I) \in \Gamma$ we denote by $\mathcal{A}^=$ its expansion by equality, i.e., the τ-structure (A, J) where $J(<) = I(<)$, $J(\perp) = I(\perp)$, and $J(=) = \{(a, a) \mid a \in A\}$. Similarly, we set $\Gamma^= = \{\mathcal{A}^= \mid \mathcal{A} \in \Gamma\}$. We call $\Gamma^=$ *negation-closed* if for every $r \in \{<, \perp, =\}$ there is a positive existential first-order formula $\varphi_r(x_1, \ldots, x_{\mathsf{ar}(r)})$ (i.e., a formula that is built up from atomic formulas using $\wedge$, $\vee$, and $\exists$) such that for all $\mathcal{A} = (A, I) \in \Gamma^=$: $A^{\mathsf{ar}(r)} \setminus I(r) = \{(a_1, \ldots, a_{\mathsf{ar}(r)}) \mid \mathcal{A} \models \varphi_r(a_1, \ldots, a_{\mathsf{ar}(r)})\}$. In other words, the complement of every relation $I(r)$ must be definable by a positive existential first-order formula.

Example 3. For any class Δ of $\{<, \perp\}$-structures such that in every $\mathcal{A} \in \Delta$, (i) $<$ is interpreted as a strict partial order and (ii) $\perp$ is interpreted as the incomparability with respect to $<$ (i.e., $x \perp y$ iff neither $x \leq y$ nor $y \leq x$), $\Delta^{=}$ is negation-closed: For every $\mathcal{A} \in \Delta^{=}$ the following equalities hold:

– $(A^2 \setminus <) = \{(x,y) \mid \mathcal{A} \models y < x \vee y = x \vee x \perp y\}$
– $(A^2 \setminus \perp) = \{(x,y) \mid \mathcal{A} \models x < y \vee x = y \vee y < x\}$
– $(A^2 \setminus =) = \{(x,y) \mid \mathcal{A} \models x < y \vee x \perp y \vee y < x\}$

In particular, the class of all semi-linear orders and all its subclasses are negation-closed (to this end, $\perp$ is part of our signature).

We say that Γ has the EHD-property (*existence of a homomorphism to a structure from Γ is* Bool(MSO, WMSO+B)*-definable*) if a Bool(MSO, WMSO+B)-sentence φ exists such that for every countable $\{<, \perp\}$-structure $\mathcal{B}$: $\mathcal{B} \models \varphi$ iff $\mathcal{B} \preceq \mathcal{A}$ for some $\mathcal{A} \in \Gamma$. The following result connects SAT-ECTL$^*(\Gamma^{=})$ with the EHD-property for the class Γ.

Proposition 4 ([5]). *Let Γ be a class of structures over $\{<, \perp\}$. If $\Gamma^{=}$ is negation-closed and Γ has the EHD-property, then* SAT-ECTL$^*(\Gamma^{=})$ *is decidable.*

In the next two sections, we show that all classes mentioned in Thm. 2 have the EHD-property. Together with Prop. 4 this implies Thm. 2.

3 Constraint ECTL* over semi-linear orders

Let Γ denote the class of all semi-linear orders (over $\{<, \perp\}$). The aim of this section is to prove that Γ has the EHD-property. For this purpose, we characterize all those structures that admit homomorphism to some element of Γ. The resulting criterion can be easily translated into WMSO. Hence, we do not need the bounding quantifier from WMSO+B here (the same will be true in the following Section 4).

It turns out that, in the case of semi-linear orders (and also ordinal forests) the existence of such a homomorphism is in fact equivalent to the existence of a *compatible expansion*. Let us fix a graph[5] $\mathcal{A} = (A, <, \perp)$. We say that $\mathcal{A}$ can be *extended* to a semi-linear order (an ordinal forest) if there is a partial order $\lhd$ such that $(A, \lhd, \perp_{\lhd})$ is a semi-linear order (a ordinal forest) *compatible* with $\mathcal{A}$, i.e.,

$$x < y \Rightarrow x \lhd y \text{ and } x \perp y \Rightarrow x \perp_{\lhd} y. \tag{1}$$

Lemma 5. *The following are equivalent for every structure $\mathcal{A} = (A, <, \perp)$:*

1. *$\mathcal{A}$ can be extended to a semi-linear order (to an ordinal forest, resp.).*
2. *$\mathcal{A} \preceq \mathcal{B}$ for some semi-linear order (ordinal tree, resp.) $\mathcal{B}$.*

The following compactness result is inspired by Wolk's work on comparability graphs of semi-linear orders [11,12]. It extends [12, Thm. 2].

[5] We call $(A, <, \perp)$ a graph to emphasize that here the binary relation symbols $<$ and $\perp$ can have arbitrary interpretations, whence we see them as two kinds of edges in an arbitrary graph.

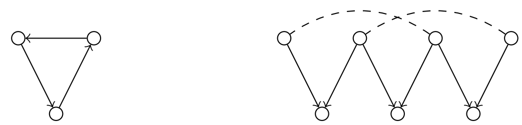

Fig. 1. A $<$-cycle of three elements and an "incomparable triple-u"; $\perp$-edges are dashed.

Lemma 6. *A structure* $\mathcal{A} = (A, <, \perp)$ *can be extended to a semi-linear order if and only if every finite substructure of* $\mathcal{A}$ *can be extended to a semi-linear order.*

Thanks to Lemma 6, given a $\{<, \perp\}$-structure $\mathcal{A}$, proving EHD only requires to look for a necessary and sufficient condition which guarantees that every finite substructure of $\mathcal{A}$ admits a homomorphism into a semi-linear order.

Given $A' \subseteq A$, we say A' is *connected (with respect to $<$)* if and only if, for all $a, a' \in A'$, there are $a_1, \ldots, a_n \in A'$ such that $a = a_1$, $a' = a_n$ and $a_i < a_{i+1}$ or $a_{i+1} < a_i$ for all $1 \leq i \leq n-1$. A *connected component* of $\mathcal{A}$ is an inclusion-maximal connected subset of A. Given a subset $A' \subseteq A$ and $c \in A'$, we say that c is a *central point of* A' if and only if for every $a \in A'$ neither $a \perp c$ nor $c \perp a$ nor $a < c$ holds. In other words, a central point of a subset $A' \subseteq A$ is a node, which has no incoming or outgoing $\perp$-edges, and no incoming $<$-edges in A'.

Example 7. A $<$-cycle (of any number of elements) does not have a central point, nor does an *incomparable triple-u*, see Figure 1. Both structures do not admit any homomorphism into a semi-linear order. While this statement is obvious for the cycle, we leave the proof for the incomparable triple-u as an exercise.

Lemma 8. *A finite structure* $\mathcal{A} = (A, <, \perp)$ *can be extended to a semi-linear order if and only if every non-empty connected* $B \subseteq A$ *has a central point.*

Let us extract the main argument for the ($\Rightarrow$)-part of the proof for later reuse:

Lemma 9. *Let* $(A, \lhd, \perp_{\lhd})$ *be a semi-linear order extending* $\mathcal{A} = (A, <, \perp)$*. If a connected subset* $B \subseteq A$ *(with respect to $<$) contains a minimal element* m *with respect to* $\lhd$*, then* m *is central in* B *(again with respect to* $\mathcal{A}$*).*

Proof. Let $b \in B$. Since B is connected, there are $b_1, \ldots, b_n \in B$ such that $b_1 = m$, $b_n = b$ and $b_i < b_{i+1}$ or $b_{i+1} < b_i$ for all $1 \leq i \leq n-1$. As $\lhd$ is compatible with $<$, this implies that $b_i \lhd b_{i+1}$ or $b_{i+1} \lhd b_i$ for all $1 \leq i \leq n-1$. Given that m is minimal, applying semi-linearity of $\lhd$, we obtain that $m = b_i$ or $m \lhd b_i$ for all $1 \leq i \leq n$. In particular, we have $m = b$ or $m \lhd b$. Since $(A, \lhd, \perp_{\lhd})$ is a semi-linear order, compatible with $(A, <, \perp)$, we cannot have $b < m$, $m \perp b$ or $b \perp m$ (since this would imply $b \lhd m$ or $m \perp_{\lhd} b$). Hence, m is central. $\square$

Proof of Lemma 8. For the direction ($\Rightarrow$) let B be any non-empty connected subset of A. Since B is finite, there is a minimal element m. Using the previous lemma we conclude that m is central in B.

We prove the direction ($\Leftarrow$) by induction on $n = |A|$. Suppose $n = 1$ and let $A = \{a\}$. The fact that $\{a\}$ has a central point implies that neither $a < a$ nor $a \perp a$ holds. Hence, $\mathcal{A}$ is a semi-linear order.

Suppose $n > 1$ and assume the statement to be true for all $i < n$. If $\mathcal{A}$ is not connected with respect to $<$, then we apply the induction hypothesis to every connected component. The union of the resulting semi-linear orders extends $\mathcal{A}$. Now assume that $\mathcal{A}$ is connected and let c be a central point of A. By the inductive hypothesis we can find $\lhd'$ such that $(A \setminus \{c\}, \lhd', \perp_{\lhd'})$ is a semi-linear order extending $\mathcal{A} \setminus \{c\}$. We define $\lhd := \lhd' \cup \{(c, x) \mid x \in A \setminus \{c\}\}$ (i.e., we add c as a smallest element), which is obviously a partial order on A.

To prove that $\lhd$ is semi-linear, let $a_1 \lhd a$ and $a_2 \lhd a$. If $a_1 = c$ or $a_2 = c$, then a_1 and a_2 are comparable by definition. Otherwise, we conclude that $a_1, a_2, a \in A \setminus \{c\}$. Hence, $a_1 \lhd' a$ and $a_2 \lhd' a$, and semi-linearity of $\lhd'$ settles the claim.

We finally show compatibility. Suppose that $a < b$. If $a = c$, then $a \lhd b$. The case $b = c$ cannot occur, because c is central in A. The remaining possibility $a \neq c \neq b$ implies that $a \lhd' b$ and hence $a \lhd b$ as desired. Finally, suppose that $a \perp b$. Then $a \neq c \neq b$, because c is central. We conclude that $a \perp_{\lhd'} b$ and also $a \perp_{\lhd} b$. ☐

We are finally ready to state the main result of this section which (together with Prop. 4) completes the proof of the first part of Theorem2:

Proposition 10. *The class of all semi-linear orders Γ has the EHD-property.*

Proof. Take $\mathcal{A} = (A, <, \perp)$. Thanks to Lemmas 5, 6 and 8, it is enough to show that WMSO can express the condition that every finite and non-empty connected substructure of $\mathcal{A}$ has a central point. This is straightforward. ☐

4 Constraint ECTL* over ordinal trees

Let Ω denote the class of all ordinal trees (over the signature $\{<, \perp\}$). The aim of this section is to prove that Ω has the EHD-property as well. We use again the notions of connected subset and central point introduced in the previous section.

Lemma 11. *Let $\mathcal{A} = (A, <, \perp)$ be a structure. There exists $\mathcal{O} \in \Omega$ with $\mathcal{A} \preceq \mathcal{O}$ if and only if every non-empty (not necessarily finite) and connected $B \subseteq A$ has a central point.*

Proof. We start with the direction ($\Rightarrow$). Due to Lemma 5 we can assume that there is a relation $\lhd$ that extends $(A, <, \perp)$ to an ordinal forest. Let $B \subseteq A$ be a non-empty connected set. Since $(A, \lhd, \perp_{\lhd})$ is an ordinal forest, B has a minimal element c with respect to $\lhd$. By Lemma 9, c is a central point of B.

For the direction ($\Leftarrow$) we first define a partition of the domain of $\mathcal{A}$ into subsets C_β for $\beta \sqsubset \chi$, where χ is an ordinal (whose cardinality is bounded by the cardinality of A). Here $\sqsubset$ denotes the natural order on ordinals. Assume that the pairwise disjoint subsets C_β have been defined for all $\beta \sqsubset \alpha$ (which is true for $\alpha = 0$ in the beginning). We define C_α as follows. Set $C_{\sqsubset \alpha} = \bigcup_{\beta \sqsubset \alpha} C_\beta \subseteq A$.

If $A \setminus C_{\sqsubset\alpha}$ is not empty, let CC_α be the set of connected components of $A \setminus C_{\sqsubset\alpha}$. Then

$$C_\alpha = \{c \in A \setminus C_{\sqsubset\alpha} \mid c \text{ is a central point of some } B \in \mathsf{CC}_\alpha\}.$$

Clearly, C_α is not empty. Hence, there is a smallest ordinal χ such that $A = C_{\sqsubset\chi}$.

For every ordinal $\alpha \sqsubset \chi$ and each element $c \in C_\alpha$ we define the sequence of connected components $\mathsf{road}(c) = (B_\beta)_{(\beta \sqsubseteq \alpha)}$, where $B_\beta \in \mathsf{CC}_\beta$ is the unique connected component with $c \in B_\beta$. This ordinal-indexed sequence keeps record of the *road* we took to reach c by storing information about the connected components to which c belongs at each stage of our process.

Given $\mathsf{road}(c) = (B_\beta)_{(\beta \sqsubseteq \alpha)}$ and $\mathsf{road}(c') = (B'_\beta)_{(\beta \sqsubseteq \alpha')}$ for some $c \in C_\alpha$ and $c' \in C_{\alpha'}$, let us define $\mathsf{road}(c) \lhd \mathsf{road}(c')$ if and only if $\alpha \sqsubset \alpha'$ and $B_\beta = B'_\beta$ for all $\beta \sqsubseteq \alpha$. This is the *prefix order* for ordinal-sized sequences of connected components.

Now let $O = \{\mathsf{road}(c) \mid c \in A\}$. Note that $\mathcal{O} = (O, \lhd, \perp_\lhd)$ is an ordinal forest, because for each $c \in C_\alpha$ the order $(\{\mathsf{road}(c') \mid \mathsf{road}(c') \unlhd \mathsf{road}(c)\}, \unlhd)$ forms the ordinal α (for each $\beta \sqsubset \alpha$ it contains exactly one road of length β).

Now we show that the mapping h with $h(c) = \mathsf{road}(c)$ is a homomorphism from $\mathcal{A}$ to $\mathcal{O}$. Take elements $a, a' \in A$ with $a \in C_\alpha$, and $a' \in C_{\alpha'}$ for some $\alpha, \alpha' \sqsubset \chi$. Let $\mathsf{road}(a) = (B_\beta)_{(\beta \sqsubseteq \alpha)}$ and $\mathsf{road}(a') = (B'_\beta)_{(\beta \sqsubseteq \alpha')}$.

If $a < a'$, then (i) $\alpha \sqsubset \alpha'$, because a' cannot be central point of a set which contains a, and (ii) $B_\beta = B'_\beta$ for all $\beta \sqsubseteq \alpha$ because a and a' belong to the same connected component of $A \setminus C_{\sqsubset\beta}$ for all $\beta \sqsubseteq \alpha$. By these observations we deduce that $\mathsf{road}(a) \lhd \mathsf{road}(a')$. If $a \perp a'$, then, without loss of generality, suppose that $\alpha \sqsubseteq \alpha'$. At stage α, a is a central point of $B_\alpha \in \mathsf{CC}_\alpha$. Since $\alpha \sqsubseteq \alpha'$, the connected component B'_α exists. We must have $B_\alpha \neq B'_\alpha$, since otherwise we would have $a \perp a' \in B_\alpha$ contradicting the fact that a is central for B_α. Therefore, $\mathsf{road}(a) \perp_\lhd \mathsf{road}(a')$.

We finally add one extra element road_0 and make this the minimal element of $\mathcal{O}$, thus finding a homomorphism from $\mathcal{A}$ into an ordinal tree. ☐

We can now complete the proof of the second part of Theorem 2

Proposition 12. *The class Ω of all ordinal trees has the* EHD*-property.*

Proof. Given a $\{<, \perp\}$-structure $\mathcal{A}$, it suffices by Lemma 11 to find an MSO-formula expressing the fact that every non-empty connected subset of $\mathcal{A}$ has a central point, which is straightforward. ☐

The procedure from the proof of Lemma 11 can be also used to embed a structure $\mathcal{A} = (A, <, \perp)$ into an ordinary tree. For this, the ordinal χ has to satisfy $\chi \leq \omega$, i.e., every element $a \in A$ has to belong to a set C_n for some finite n. We use this observation in Section 5. Unfortunately, our results from Section 5 show that the condition $\chi \leq \omega$ cannot be expressed in Bool(MSO, WMSO+B). On the other hand, by unfolding the above fixpoint procedure for h steps (for a fixed $h \in \mathbb{N}$), we obtain an MSO-formula that expresses the existence of a homomorphism into a tree of heigth h. This shows (3) from Thm. 2. Details can be found in the long version [6].

5 Trees do not have the EHD-property

Let Θ be the class of all countable trees (over $\{<, \perp\}$). In this section, we prove that the logic $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO}{+}\mathsf{B})$ cannot distinguish between graphs that admit a homomorphism to some element of Θ and those that do not. Thus, Θ does not have the EHD-property proving our second main result Thm. 1.

Heading for a contradiction, assume that φ is a sentence such that a countable structure $\mathcal{A} = (A, <, \perp)$ satisfies φ if and only if there is a homomorphism from $\mathcal{A}$ to some $\mathcal{T} \in \Theta$. Let k be the quantifier rank of φ. We construct two graphs $\mathcal{E}_k$ and $\mathcal{U}_k$ such that $\mathcal{E}_k$ admits a homomorphism into a tree while $\mathcal{U}_k$ does not. We then use the Ehrenfeucht-Fraïssé game for $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO}{+}\mathsf{B})$ to show that φ cannot separate these two structures, contradicting our assumption.

5.1 The WMSO+B-Ehrenfeucht-Fraïssé-game

The k-round WMSO+B-Ehrenfeucht-Fraïssé-game (*k-round game* in the following) on a pair of structures $(\mathcal{A}, \mathcal{B})$ over the same finite relational signature σ is played by spoiler and duplicator as follows.[6] In the following, A denotes the domain of $\mathcal{A}$ and B the domain of $\mathcal{B}$.

The game starts in position $p_0 = (\mathcal{A}, \emptyset, \emptyset, \mathcal{B}, \emptyset, \emptyset)$. In general, before playing the i-th round (for $1 \leq i \leq k$) the game is in a position

$$p = (\mathcal{A}, a_1, \dots, a_{i_1}, A_1, \dots, A_{i_2}, \mathcal{B}, b_1, \dots, b_{i_1}, B_1, \dots, B_{i_2}), \tag{2}$$

where $i_1 + i_2 = i - 1$, $a_j \in A$ and $b_j \in B$ for all $1 \leq j \leq i_1$, and $A_j \subseteq A$ and $B_j \subseteq B$ are a finite sets for all $1 \leq j \leq i_2$.

In the i-th round spoiler and duplicator produce the next position as follows. Spoiler chooses to play one of the following three possibilities: either he plays an element move or a set move like in the usual WMSO-game (see [7]), or a *Bound move*, in which spoiler first chooses one of the structures $\mathcal{A}$ or $\mathcal{B}$ and a natural number $l \in \mathbb{N}$. Duplicator responds with another number $m \in \mathbb{N}$. Then the game continues as in the case of a set move with the restrictions that spoiler has to choose a subset of size at least m from his chosen structure and duplicator has to respond with a set of size at least l.

After k rounds, the game ends in a position

$$p = (\mathcal{A}, a_1, \dots, a_{i_1}, A_1, \dots, A_{i_2}, \mathcal{B}, b_1, \dots, b_{i_1}, B_1, \dots, B_{i_2}).$$

Duplicator wins the game if

1. $a_j \in A_k \Leftrightarrow b_j \in B_k$ for all $1 \leq j \leq i_1$ and all $1 \leq k \leq i_2$,
2. $a_j = a_k \Leftrightarrow b_j = b_k$ for all $1 \leq j < k \leq i_1$, and
3. for all relation symbols $R \in \sigma$ (of arity n) and all $j_1, j_2, \dots, j_n \in \{1, \dots, i_1\}$, $(a_{j_1}, \dots, a_{j_n}) \in R^{\mathcal{A}}$ iff $(b_{j_1}, \dots, b_{j_n}) \in R^{\mathcal{B}}$.

[6] For the ease of presentation we assume that $\mathcal{A}$ and $\mathcal{B}$ are infinite structures.

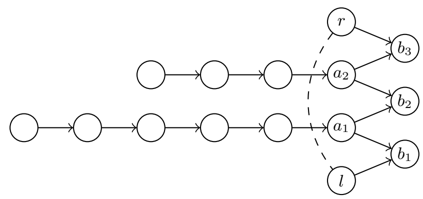

Fig. 2. The standard $(5,3)$-triple-u, where we only draw the Hasse diagram for $<^D$, and where dashed edges are $\perp$-edges.

As one expects, the k-round game is closely connected to definability with WMSO+B-formulas of quantifier rank k: If p is a position as in (2), the structures $(\mathcal{A}, a_1, \dots, a_{i_1}, A_1, \dots, A_{i_2})$ and $(\mathcal{B}, b_1, \dots, b_{i_1}, B_1, \dots, B_{i_2})$ are indistinguishable by all WMSO+B-formulas of quantifier rank k if and only if duplicator has a winning strategy in the k-round WMSO+B-EF-game started in p.

5.2 The Embeddable and the Unembeddable Triple-U-Structures

In this section we define for every $k \geq 0$ structures $\mathcal{E}_k$ and $\mathcal{U}_k$ with the following properties: (i) $\mathcal{E}_k$ can be mapped homomorphically into a tree, whereas $\mathcal{U}_k$ cannot, and (ii) duplicator wins the k-round EF-game for both WMSO+B and MSO on $(\mathcal{E}_k, \mathcal{U}_k)$.

The *standard plain triple-u* is the structure $\mathcal{P} = (P, <, \perp)$, where

- $P = \{l, r, a_1, a_2, b_1, b_2, b_3\}$,
- $<= \{(l, b_1), (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), (r, b_3)\}$, and
- $\perp= \{(l, r), (r, l)\}$.

For $n, m \in \mathbb{N}$, the *standard (n,m)-triple-u* is the structure $\mathcal{G}_{n,m} = (D, <, \perp)$, where $D = \{l, r, a_1, a_2, b_1, b_2, b_3\} \cup (\{1, 2, \dots, n\} \times \{a_1\}) \cup (\{1, 2, \dots, m\} \times \{a_2\})$, and $<, \perp$ are the minimal relations such that $<$ is transitive and

- $\mathcal{G}_{n,m}$ restricted to $\{l, r, a_1, a_2, b_1, b_2, b_3\}$ is the standard plain triple-u, and
- $(a_1, 1) < (a_1, 2) < \cdots < (a_1, n) < a_1$, $(a_2, 1) < (a_2, 2) < \cdots < (a_2, m) < a_2$.

We call a structure $(V, <, \perp)$ a *plain triple-u* (resp. *(n,m)-triple-u*) if it is isomorphic to the standard plain triple-u (resp., standard (n,m)-triple-u). Fig. 2 depicts a $(5,3)$-triple-u.

For all $n, m \in \mathbb{N}$ and each (n,m)-triple-u $\mathcal{W}$ we fix an isomorphism $\psi_{\mathcal{W}}$ between $\mathcal{W}$ and the standard (n,m)-triple-u. This isomorphism is unique if $n \neq m$. If $n = m$, there is an automorphism of $\mathcal{G}_{n,n}$ exchanging the nodes l and r. Thus, choosing $\psi_{\mathcal{W}}$ means to fix the left node of the triple-u. For $x \in \{l, r, a_1, a_2, b_1, b_2, b_3\}$ we write $\mathcal{W}.x$ for the node $w \in \mathcal{W}$ such that $\psi_{\mathcal{W}}(w) = x$.

Let $k \in \mathbb{N}$ be a natural number. Fix a strictly increasing sequence $(n_{k,i})_{i \in \mathbb{N}}$ such that the linear order of length $n_{k,i}$ and the linear order of length $n_{k,j}$ are equivalent with respect to WMSO+B-formulas of quantifier rank up to k for all $i, j \in \mathbb{N}$. Such a sequence exists because there are (up to equivalence) only finitely many $\mathsf{WMSO{+}B}$-formulas of quantifier rank k. Since the linear orders of length $n_{k,i}$ are finite, they are equivalent with respect to both MSO-formulas and WMSO-formulas of quantifier rank up to k. Using these linear orders, we define two structures:

Let $\mathcal{E}_k$ (for embeddable) be the structure that consists of

1. the disjoint union of $\aleph_0$ many $(n_{k,1}, n_{k,j})$-triple-u's and $\aleph_0$ many $(n_{k,j}, n_{k,1})$-triple-u's for all $j \geq 2$, and
2. one extra node d, and for each triple-u $\mathcal{W}$ from 1. a $<$-edge from $\mathcal{W}.l$ to d.

The structure $\mathcal{U}_k$ (for unembeddable) is defined in the same way, except that in 1. we take the disjoint union of $\aleph_0$ many $(n_{k,j}, n_{k,j})$-triple-u's for all $j \geq 2$. The following lemma can be shown using the procedure on the central points from the ordinal tree setting described in the proof of Lemma 11.

Lemma 13. *For all $k \in \mathbb{N}$, $\mathcal{E}_k$ admits a homomorphism to a tree, whereas $\mathcal{U}_k$ does not admit a homomorphism to a tree.*

We prove that Θ does not have the EHD-property by showing that duplicator wins the k-round MSO-EF-game and the $\mathsf{WMSO{+}B}$-EF-game on the pair $(\mathcal{E}_k, \mathcal{U}_k)$ for each $k \in \mathbb{N}$. Hence, the two structures are not distinguishable by $\mathsf{Bool}(\mathsf{MSO}, \mathsf{WMSO{+}B})$-formulas of quantifier rank k. For MSO this is rather simple. Since the linear orders of length $n_{k,i}$ and $n_{k,j}$ are indistinguishable up to quantifier rank k, it is straightforward to compose the strategies on these pairs of paths to a strategy on the whole structures for the k-round game. It is basically the same proof as the one showing that a strategy on a pair $(\biguplus_{i\in I} \mathcal{A}_i, \biguplus_{i\in I} \mathcal{B}_i)$ of disjoint unions can be composed from strategies on the pairs $(\mathcal{A}_i, \mathcal{B}_i)$.

Composing local strategies to a global strategy in the $\mathsf{WMSO{+}B}$-EF-game is more difficult because strategies are not closed under infinite disjoint unions. For instance, let $\mathcal{A}$ be the disjoint union of infinitely many copies of the linear order of size $n_{k,1}$ and $\mathcal{B}$ be the disjoint union of all linear orders of size $n_{k,j}$ for all $j \in \mathbb{N}$. Clearly, duplicator has a winning strategy in the k-round game starting on the pair that consists of the linear order of size $n_{k,1}$ and the linear order of size $n_{k,j}$. But in $\mathcal{A}$ every linear suborder has size bounded by $n_{k,1}$, while $\mathcal{B}$ has linear suborders of arbitrary finite size. This difference is expressible in $\mathsf{WMSO{+}B}$. Nevertheless, composition of local strategies to a global strategy on disjoint unions $\mathcal{A} = \biguplus_{n\in\mathbb{N}} \mathcal{A}_n$ and, $\mathcal{B} = \biguplus_{n\in\mathbb{N}} \mathcal{B}_n$ works if we pose two restrictions:

1. $\mathcal{A}_n$ and $\mathcal{B}_n$ are finite for all $n \in \mathbb{N}$.
2. For each $n \in \mathbb{N}$ duplicator has a strategy in the game on $(\mathcal{A}_n, \mathcal{B}_n)$ that *preserves a first big set* in the sense that there is a $c \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have: If spoiler starts the WMSO-EF-game on $(\mathcal{A}_n, \mathcal{B}_n)$ with a set move choosing a set of size m in $\mathcal{A}_n$ or $\mathcal{B}_n$, then duplicator answers with a set of size at least $\frac{m}{c}$.

Under these conditions, duplicator has the following strategy for bound moves in the game on $(\mathcal{A}, \mathcal{B})$: If spoiler chooses w.l.o.g structure $\mathcal{A}$ and bound $l \in \mathbb{N}$, duplicator chooses the number $m_1 + m_2$ where m_1 is the sum of all the elements of all parts $\mathcal{A}_i$ in which elements or sets have been chosen before and $m_2 = c{\cdot}l$ where c is the constant denoted above. This forces spoiler to choose m_2 many elements in fresh parts of $\mathcal{A}$ Thus, the first big set preserving strategies allow duplicator to choose at least $\frac{m_2}{c} = l$ elements in corresponding fresh parts of $\mathcal{B}$. Using a variant of this composition result where we choose the pair $(\mathcal{A}_n, \mathcal{B}_n)$ of the union dynamically to be $(\mathcal{G}_{n_{k,1},n_{k,j}}, \mathcal{G}_{n_{k,j},n_{k,j}})$ or $(\mathcal{G}_{n_{k,j},n_{k,1}}, \mathcal{G}_{n_{k,j},n_{k,j}})$ (depending on spoiler's moves) we can prove the following result.

Proposition 14. *For every k, duplicator has a winning strategy in the k-round* WMSO+B*-EF-game on $(\mathcal{E}_k, \mathcal{U}_k)$. Hence, Θ does not have the* EHD*-property.*

6 Open problems

The main open problem that remains is whether the problem SAT-ECTL$^*(\Theta^=)$ is decidable for the class Θ of all trees (or equivalently, the single infinite binary tree). We have only proved that the EHD-method cannot yield decidability. Currently, we are investigating automata theoretic approaches to this question.

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