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# Satisfiability of ECTL* with local tree constraints

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**Abstract** Recently, we have shown that satisfiability for the temporal logic ECTL* with local constraints over $(\mathbb{Z}, <, =)$ is decidable using a new technique [7]. This approach reduces the satisfiability problem of ECTL* with constraints over some structure $\mathcal{A}$ (or class of structures) to the problem whether $\mathcal{A}$ has a certain model theoretic property that we called EHD (for "existence of homomorphisms is definable"). Here we apply this approach to structures that are tree-like and obtain several results. We show that satisfiability of ECTL* with constraints is decidable over (i) semi-linear orders (i.e., tree-like structures where branches form arbitrary linear orders), (ii) ordinal trees (semi-linear orders where the branches form ordinals), and (iii) infinitely branching trees of height $h$ for each fixed $h \in \mathbb{N}$. We prove that all these classes of structures have the property EHD. In contrast, we introduce Ehrenfeucht-Fraïssé-games for WMSO+B (weak MSO with the bounding quantifier) and use them to show that the infinite (order) tree does not have the EHD-property. As a consequence, our technique cannot be used to establish whether satisfiability of ECTL* with constraints over the infinite (order) tree is decidable. A preliminary version of this paper has appeared as [6].

**Keywords** temporal logics · ECTL* · concrete domains · local constraints · semi-linear orders · ordinal trees · WMSO+B · EF-games

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## 1 Introduction

Temporal logics like LTL, CTL or CTL* are nowadays standard languages for specifying system properties in verification. These logics are interpreted over node labeled graphs (Kripke structures), where the node labels (also called atomic propositions) represent abstract properties of a system. In the last ten years, many extensions of temporal logics with constraints have been proposed in order to address the need to express more than just abstract properties [4,10,11,12]. In this setting, a model of a formula is not only a Kripke structure but a Kripke structure where to every node several values from some fixed structure $\mathcal{C}$ (called a concrete domain) are assigned. The logic is then enriched in such a way that it has access to the relations of the concrete domain. For instance, if $\mathcal{C} = (\mathbb{Z}, <, =)$ then every node of the Kripke structure holds several integers and the logic can express that the values assigned to neighboring nodes are in one of the relations $=$ or $<$. Such statements are called local constraints.

While for LTL with constraints over $(\mathbb{Z}, <, =)$ satisfiability was proven to be decidable and PSPACE-complete [10], the problem remained open for its branching time version CTL* with constraints. In our paper [7], we introduced a new method (called EHD-method in the following) which shows decidability of the satisfiability problem for CTL* extended by local constraints over the concrete domain $(\mathbb{Z}, <, =)$. In [8] we extended the EHD-method to *extended computation tree logic* (ECTL*) with constraints over the integers (a powerful temporal logic that properly extends CTL*) and proved that satisfiability is still decidable. This result greatly improves the partial results on fragments of CTL* obtained by Bozzelli, Gascon and Pinchinat [4,5,15].

EHD stands for "the existence of a homomorphism is definable" since our method connects satisfiability of ECTL* with constraints over $\mathcal{C}$ with the definability of homomorphism to $\mathcal{C}$. The idea can be summarized as follows. Let $\mathcal{C}$ be any concrete domain over a relational signature $\sigma$. We then proved in [7] that satisfiability of ECTL* with constraints over $\mathcal{C}$ is decidable if $\mathcal{C}$ has the following two properties:

1. The structure $\mathcal{C}$ is negation-closed, i.e., the complement of any relation $R \in \sigma$ is definable in positive existential first-order logic.
2. There exists a sentence $\varphi$ of the logic BMWB (explained below), which defines the existence of a homomorphism to $\mathcal{C}$ in the following sense: For any countable $\sigma$-structure $\mathcal{A}$ there is a homomorphism from $\mathcal{A}$ to $\mathcal{C}$ if and only if $\mathcal{A} \models \varphi$.

With BMWB we denote the set of all Boolean combinations of MSO-formulas (monadic second-order logic) and WMSO+B-formulas, where WMSO+B is weak monadic second-order logic with the bounding quantifier. The bounding quantifier allows to express that there is a bound on the size of finite sets satisfying a certain property.

The reason for choosing BMWB to characterize the existence of a homomorphism to the concrete domain is that it is the most expressive logic (that we are aware of) which enjoys all properties that we need to apply the EHD-method:

1. Decidability of the satisfiability problem over the class of infinite node-labeled trees,
2. closure under boolean combinations with MSO, and
3. compatibility with one dimensional first-order interpretations and with the $k$-copy operation, see [8].

Satisfiability of WMSO+B over infinite node-labeled trees was shown to be decidable in [3] (in contrast, decidability of full monadic second-order logic with the bounding quantifier over words is undecidable [1]).

In [7] we proved that the existence of a homomorphism to $(\mathbb{Z}, <, =)$ can be expressed in BMWB (or even in WMSO+B). Since $(\mathbb{Z}, <, =)$ is negation closed, the EHD-method yields decidability of the satisfiability problem for $\mathsf{ECTL}^*$ with constraints over $(\mathbb{Z}, <, =)$.

The question comes naturally, whether the EHD-method can also be applied to other concrete domains. An interesting candidate in this setting is the full infinitely branching infinite (order) tree $\mathcal{T}_\infty = (\mathbb{N}^*, <, \perp, =)$, where $<$ denotes the prefix order on $\mathbb{N}^*$ and $\perp$ denotes the incomparability relation with respect to $<$ (the incomparability relation $\perp$ is added in order to obtain negation closure). Unfortunately, this hope is disappointed by one of the main results of this work:

**Theorem 1** *There is no* BMWB*-sentence $\psi$ such that for every countable structure $\mathcal{A}$ (over the signature $\{<, \perp, =\}$) we have: $\mathcal{A} \models \psi$ if and only if there is a homomorphism from $\mathcal{A}$ to $\mathcal{T}_\infty$.*

This result is shown using a suitable Ehrenfeucht-Fraïssé-game.

Theorem 1 shows that the EHD-method cannot be applied to the concrete domain $\mathcal{T}_\infty$. Of course, this does not imply that satisfiability for $\mathsf{ECTL}^*$ with constraints over $\mathcal{T}_\infty$ is undecidable. In fact, a recent work from Demri and Deters [9] established decidability of satisfiability for $\mathsf{CTL}^*$ with constraints over $\mathcal{T}_\infty$, and PSPACE-completeness of the corresponding LTL-fragment.

Nonetheless we fruitfully apply the EHD-method to other tree-like structures, such as semi-linear orders, ordinal trees, and infinitely branching trees of a fixed height. *Semi-linear orders* are partial orders that are tree-like in the sense that for every element $x$ the set of all smaller elements form a linear suborder. If this linear suborder is an ordinal (for every $x$) then one has an *ordinal tree*. Ordinal trees are widely studied in descriptive set theory and recursion theory. Note that a tree is a semi-linear order with a unique minimal element, where for every element the set of all smaller elements is a finite linear order.

In the integer-setting from [7,8], we investigated satisfiability of $\mathsf{ECTL}^*$-formulas with constraints over one fixed structure (integers with additional relations). For semi-linear orders and ordinal trees it is more natural to consider satisfiability with respect to a class of concrete domains $\Gamma$ (over a fixed signature $\sigma$): The question becomes, whether for a given constraint $\mathsf{ECTL}^*$-formula $\varphi$ there is a concrete domain $\mathcal{C} \in \Gamma$ such that $\varphi$ is satisfiable by some model with concrete values from $\mathcal{C}$. If a class $\Gamma$ has a universal structure[1] $\mathcal{U}$, then satisfiability with respect to the class $\Gamma$ is equivalent to satisfiability with respect to $\mathcal{U}$ because one easily shows that a formula $\varphi$ has a model with some concrete domain from $\Gamma$ if and only if it has a model with concrete domain $\mathcal{U}$. A typical class with a universal model is the one of all countable linear orders, for which $(\mathbb{Q}, <)$ is universal. Similarly, for the class of all countable trees the infinitely branching infinite tree as well as the binary infinite tree are universal. In the appendix we construct a universal countable semi-linear order. Since this particular universal structure appears to be less natural than $(\mathbb{Q}, <)$ or the infinite binary tree, we have decided to formulate our decidability result for the

[1] A structure $\mathcal{U}$ is universal for a class $\Gamma$ if there is a homomorphic embedding of every structure from $\Gamma$ into $\mathcal{U}$ and $\mathcal{U}$ belongs to $\Gamma$.

class of all semi-linear orders. Moreover, there is no universal structure for the class of countable ordinal trees (for a similar reason as the one showing that the class of countable ordinals does not contain a universal structure).

Application of the EHD-method to semi-linear orders and ordinal trees gives the following decidability results.

**Theorem 2** *Satisfiability of* $\mathsf{ECTL}^*$*-formulas with constraints over each of the following classes is decidable:*

*(1) the class of all semi-linear orders,*
*(2) the class of all ordinal trees, and*
*(3) for each* $h \in \mathbb{N}$*, the class of all order trees of height* $h$*.*

Concerning computational complexity, let us remark that in [7,8] we did not present an upper bound on the complexity of our decision procedure. The reason for this is that the authors of [3] do not prove an upper bound for the complexity of satisfiability of WMSO+B over infinite trees, even in the case that the input formula has bounded quantifier depth (and it is not clear how to obtain such a bound from the proof of [3]). Here, the situation is slightly different. Our applications of the EHD-method for the proof Theorem 2 do not need the bounding quantifier, and classical WMSO (for semi-linear orders) and MSO (for ordinal trees and trees of bounded height) suffice. Moreover, the formulas that express the existence of a homomorphism have only small quantifier depth (at least for semi-linear orders and ordinal trees; for trees of bounded height, the quantifier depth depends on the height). This fact can be exploited and yields a triply exponential upper bound on the time complexity in (1) and (2) from Theorem 2 (this bound does not match the doubly exponential lower bound inherited from the satisfiability problem of $\mathsf{ECTL}^*$ without constraints). We skipped the proof details, since we conjecture the exact complexity to be doubly exponential.

The paper is organized as follows: In Section 2 we introduce the necessary machinery concerning Kripke structures, tree-like partial orders and the logics MSO, WMSO+B and $\mathsf{ECTL}^*$ with constraints. Moreover, we explain the EHD-method (more details can be found in [8]). Using the EHD-method, Theorem 2 is proved in Sections 3 (for semi-linear orders), 4 (for ordinal trees) and 5 (for trees of bounded height $h$). Section 6 introduces an Ehrenfeucht-Fraïssé-game for WMSO+B and uses this game to prove Theorem 1.

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## 2 Preliminaries

In this section we introduce basic notations concerning Kripke structures, various classes of tree-like structures, and the logics MSO, WMSO+B, and $\mathsf{ECTL}^*$ with constraints.

### 2.1 Structures

Let $\mathsf{P}$ be a countable set of (atomic) propositions. A Kripke structure (over $\mathsf{P}$) is a triple $\mathcal{K} = (D, \to, \rho)$, where:

- $D$ is an arbitrary set of nodes (or states),
- $\rightarrow$ is a binary relation on $D$ such that for all $u \in D$ there exists $v \in D$ with $u \rightarrow v$, and
- $\rho : D \rightarrow 2^{\mathsf{P}}$ is a labeling function that assigns to every node a finite set of atomic propositions.

A (finite relational) signature is a finite set $\sigma$ of relation symbols. Every relation symbol $R \in \sigma$ has an associated arity $\mathsf{ar}(R) \geq 1$. A $\sigma$-structure is a pair $\mathcal{A} = (A, I)$, where $A$ is a non-empty set and $I$ maps every $R \in \sigma$ to an $\mathsf{ar}(R)$-ary relation over $A$. A simple example of $\{<\}$-structure is $(\mathbb{Z}, I)$, where $I(<)$ is, as expected, $\{(a, b) \in \mathbb{Z}^2 \mid a < b\}$. Quite often, we will identify the relation $I(R)$ with the relation symbol $R$, and we will specify a $\sigma$-structure as $(A, R_1, \ldots, R_n)$, where $\sigma = \{R_1, \ldots, R_n\}$. In the example above, we would simply write $(\mathbb{Z}, <)$ for $(\mathbb{Z}, I)$.

Given $\mathcal{A} = (A, R_1, \ldots, R_n)$ and given a subset $B$ of $A$, we define $R_{i\restriction B} = R_i \cap B^{\mathsf{ar}(R_i)}$ and $\mathcal{A}_{\restriction B} = (B, R_{1\restriction B}, \ldots, R_{n\restriction B})$ (the *restriction of $\mathcal{A}$ to the set $B$*).

For a subsignature $\tau \subseteq \sigma$, a $\tau$-structure $\mathcal{B} = (B, J)$ and a $\sigma$-structure $\mathcal{A} = (A, I)$, a *homomorphism* from $\mathcal{B}$ to $\mathcal{A}$ is a mapping $h : B \rightarrow A$ such that for all $R \in \tau$ and all tuples $(b_1, \ldots, b_{\mathsf{ar}(R)}) \in J(R)$ we have $(h(b_1), \ldots, h(b_{\mathsf{ar}(R)})) \in I(R)$. We write $\mathcal{B} \preceq \mathcal{A}$ if there is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$. Note that we do not require this homomorphism to be injective.

We now introduce *decorated* Kripke structures. These will be the structures on which we interpret constraint $\mathsf{ECTL}^*$-formulas. Fix a $\sigma$-structure $\mathcal{C} = (C, I)$, the *concrete domain*, and let $\mathsf{Reg}$ be a countable set of register variables.

**Definition 3** A $\mathcal{C}$-*decorated* Kripke structure $\mathbb{K}$ is a tuple $(D, \rightarrow, \rho, \gamma)$ where:

- $(D, \rightarrow, \rho)$ is a Kripke structure, and
- $\gamma : D \times \mathsf{Reg} \rightarrow C$ is a *valuation function*, assigning a value from the concrete domain to each variable from $\mathsf{Reg}$ in each node of the Kripke structure.

We can imagine a $\mathcal{C}$-decorated Kripke structure as a directed graph, where in addition we assign to each node $v$ a finite set of atomic propositions and an infinite tuple $(a_1, a_2, \dots)$ of values from the concrete domain $\mathcal{C}$, where $a_i$ is the value of the register variable $r_i \in \mathsf{Reg} = \{r_1, r_2, \ldots\}$ at node $v$.

**Definition 4** A $\mathcal{C}$-decorated Kripke path $\mathbb{P}$ is a $\mathcal{C}$-decorated Kripke structure of the form $\mathbb{P} = (\mathbb{N}, \mathsf{S}, \rho, \gamma)$ where $\mathsf{S}$ is the successor relation on $\mathbb{N}$.

## 2.2 Tree-like structures

We now introduce trees in the sense of Wolk [21], which are also known as *semi-linear orders*. They are partial orders $\mathcal{P} = (P, <)$ with the additional property that for all $p \in P$ the suborder induced by $\{p' \in P \mid p' \leq p\}$ forms a linear order. This property is equivalent to the one formulated by Wolk [21]: Given incomparable elements $p_1, p_2 \in P$, there is no $q \in P$ such that $p_1 < q$ and $p_2 < q$, i.e., two incomparable elements cannot have a common descendant. Clearly all trees (in the usual sense) satisfy this property, but not vice-versa.

We call a semi-linear order $\mathcal{P} = (P, <)$ an *ordinal forest* (resp., *forest*) if for all $p \in P$ the suborder induced by $\{p' \in P \mid p' \leq p\}$ is an ordinal (resp., a finite linear order). A (ordinal) forest is a *(ordinal) tree* if it has a unique minimal element. A forest $\mathcal{F}$ of *height* $h$ (for $h \in \mathbb{N}$) is a forest that contains a linear suborder with $h+1$ many elements but no linear suborder with $h+2$ elements. We say that an element $x \in P$ is *at level* $i$ if $|\{y \in P \mid y < x\}| = i$. Thus, every minimal element is at level 0.

Given a partial order $(P, <)$, we denote by $\perp_<$ the *incomparability relation* defined by $p \perp_< q$ if and only if neither $p \leq q$ nor $q \leq p$. Given a $\{<, \perp, =\}$-structure $\mathcal{P} = (P, <, \perp, =)$ such that $(P, <)$ is a semi-linear order (resp., ordinal tree, tree of height $h$), $=$ is the equality relation on $P$, and $\perp = \perp_<$, then we also say that $\mathcal{P}$ is a semi-linear order (resp. ordinal tree, tree of height $h$).

Let us mention that the class of all countable semi-linear orders contains a universal structure (see Appendix A). On the other hand, the class of all countable ordinal trees does not contain a universal structure, but there is a fixed uncountable ordinal tree such that all countable ordinal trees embed into this uncountable ordinal tree.

### 2.3 Logics

*Monadic second-order logic* (MSO) is the extension of first-order logic where also quantification over subsets of the universe is allowed. Let us fix a countably infinite set $\mathsf{Var}_1$ of first-order variables that range over elements of a structure and a countably infinite set $\mathsf{Var}_2$ of second-order variables that range over subsets of a structure. MSO-formulas over the signature $\sigma$ are given by the following grammar, where $R \in \sigma$, $x, y, x_1, \ldots, x_{\mathsf{ar}(R)} \in \mathsf{Var}_1$, and $X \in \mathsf{Var}_2$:

$$\varphi ::= R(x_1, \ldots, x_{\mathsf{ar}(R)}) \mid x = y \mid x \in X \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\, \varphi \mid \exists X\, \varphi.$$

*Weak monadic second-order logic* (WMSO) has the same syntax as MSO but second-order variables only range over finite subsets of the underlying structure.

Finally, WMSO+B is the extension of WMSO by the *bounding quantifier* $\mathsf{B}X\,\varphi$ (see [2]) whose semantics is given by $\mathcal{A} \models \mathsf{B}X\,\varphi(X)$ (with $\mathcal{A} = (A, I)$) if and only if there is a bound $b \in \mathbb{N}$ such that $|B| \leq b$ for every finite subset $B \subseteq A$ with $\mathcal{A} \models \varphi(B)$. The *quantifier rank* of a WMSO+B-formula is the maximal number of nested quantifiers (existential, universal, and bounding quantifiers) in the formula. With Bool(MSO, WMSO+B), in short BMWB, we denote the set of all boolean combinations of MSO-formulas and WMSO+B-formulas.

*Extended computation tree logic* ($\mathsf{ECTL}^*$) is a branching time temporal logic first introduced in [18,20] as an extension of $\mathsf{CTL}^*$. As the latter, $\mathsf{ECTL}^*$ is interpreted on Kripke structures, but while $\mathsf{CTL}^*$ allows to specify LTL properties of infinite paths of such models, $\mathsf{ECTL}^*$ can describe regular (i.e., MSO-definable) properties of paths. In [8] we introduced an extension of $\mathsf{ECTL}^*$, called constraint $\mathsf{ECTL}^*$, which enriches $\mathsf{ECTL}^*$ by local constraints in path formulas.

We now recall the definition of constraint path MSO-formulas, which take the role of path formulas in constraint $\mathsf{ECTL}^*$. First we consider MSO for colored infinite paths (also known as $\omega$-words). This is MSO as introduced at the beginning of Section 2.3 where the signature $\sigma$ is set to $\mathsf{P} \cup \{\mathsf{S}\}$, where $\mathsf{S}$ is the successor relation and the atomic propositions from $\mathsf{P}$ are seen as unary predicates. This logic is also known as S1S (see [19]). To obtain *constraint path* MSO (with constraints over a signature $\tau$), denoted as $\mathsf{MSO}(\tau)$, we extend S1S by atomic formulas that describe local constraints over the concrete domain. Thus, given a $\tau$-structure $\mathcal{C}$, $\mathsf{MSO}(\tau)$ can be interpreted over the class of $\mathcal{C}$-decorated Kripke paths.

In this paper we exclusively consider tree-like concrete domains over the fixed signature $\tau = \{<, \perp, =\}$. Therefore, we simplify the presentation and introduce constraint path $\mathsf{MSO}$ only over $\tau$. For a more general presentation we refer the reader to [8]. Fix a set $\mathsf{P}$ of atomic propositions and a set $\mathsf{Reg}$ of register variables. $\mathsf{MSO}(\tau)$-formulas are defined by the following grammar:

$$\psi \ ::= \ p(x) \mid x = \mathsf{S}(y) \mid x \in X \mid \neg\psi \mid (\psi \wedge \psi) \mid \exists x\, \psi \mid \exists X\, \psi \mid (\mathsf{S}^i r_1 \sim \mathsf{S}^j r_2)(x)$$

where $p \in \mathsf{P}$, $x, y \in \mathsf{Var}_1$, $X \in \mathsf{Var}_2$, $\sim \in \tau$, $i, j \in \mathbb{N}$ and $r_1, r_2 \in \mathsf{Reg}$. We call formulas of the form $(\mathsf{S}^i r_1 \sim \mathsf{S}^j r_2)(x)$ *atomic constraints*. Note that we use the successor function $\mathsf{S}$ instead of the binary successor relation, which simplifies the technical details.

Let $\mathbb{P} = (\mathbb{N}, \mathsf{S}, \rho, \gamma)$ be a $\mathcal{C}$-decorated Kripke path, where $\mathcal{C} = (C, <, \perp, =)$ is a tree-like concrete domain. Let $\eta : (\mathsf{Var}_1 \cup \mathsf{Var}_2) \to (\mathbb{N} \cup 2^{\mathbb{N}})$ be a valuation function mapping first-order variables to natural numbers and second-order variables to sets of natural numbers. The satisfaction relation $\models$ is defined by structural induction as follows:

$(\mathbb{P}, \eta) \models p(x)$ iff $p \in \rho(\eta(x))$.
$(\mathbb{P}, \eta) \models x = \mathsf{S}(y)$ iff $\eta(x) = \eta(y) + 1$.
$(\mathbb{P}, \eta) \models x \in X$ iff $\eta(x) \in \eta(X)$.
$(\mathbb{P}, \eta) \models \neg\psi$ iff it is not the case that $(\mathbb{P}, \eta) \models \psi$.
$(\mathbb{P}, \eta) \models (\psi_1 \wedge \psi_2)$ iff $(\mathbb{P}, \eta) \models \psi_1$ and $(\mathbb{P}, \eta) \models \psi_2$.
$(\mathbb{P}, \eta) \models \exists x\, \psi$ iff there is an $n \in \mathbb{N}$ such that $(\mathbb{P}, \eta[x \mapsto n]) \models \psi$.
$(\mathbb{P}, \eta) \models \exists X\, \psi$ iff there is an $E \subseteq \mathbb{N}$ such that $(\mathbb{P}, \eta[X \mapsto E]) \models \psi$.
$(\mathbb{P}, \eta) \models (\mathsf{S}^i r_1 \sim \mathsf{S}^j r_2)(x)$ iff $\gamma(\eta(x) + i, r_1) \sim \gamma(\eta(x) + j, r_2)$ in $\mathcal{C}$.

Notice how the term $\mathsf{S}^i r$, inside an atomic constraint, is used to indicate that we are referring to the value of the register variable $r$ in the $i$-th successor position of the current one.

For an $\mathsf{MSO}(\tau)$-formula $\psi$ the satisfaction relation only depends on the variables occurring freely in $\psi$. This motivates the following notation: If $\psi(X_1, \ldots, X_m)$ is an $\mathsf{MSO}(\tau)$-formula where $X_1, \ldots, X_m \in \mathsf{Var}_2$ are the only free variables, we write $\mathbb{P} \models \psi(A_1, \ldots, A_m)$ if and only if, for every valuation function $\eta$ such that $\eta(X_i) = A_i$, we have $(\mathbb{P}, \eta) \models \psi$.

Having defined $\mathsf{MSO}(\tau)$-formulas we are ready to define constraint $\mathsf{ECTL}^*$ over the signature $\tau$, denoted by $\mathsf{ECTL}^*(\tau)$:

$$\varphi ::= \mathsf{E}\psi(\underbrace{\varphi, \ldots, \varphi}_{m \text{ times}}) \mid (\varphi \wedge \varphi) \mid \neg\varphi \tag{1}$$

where $\psi(X_1, \ldots, X_m)$ is an $\mathsf{MSO}(\tau)$-formula in which at most the second-order variables $X_1, \ldots, X_m \in \mathsf{Var}_2$ are allowed to occur freely. Note that for $m = 0$ we get atomic $\mathsf{ECTL}^*(\tau)$-formulas.

$\mathsf{ECTL}^*(\tau)$-formulas are evaluated over nodes of $\mathcal{C}$-decorated Kripke structures. Let $\mathbb{K} = (D, \to, \rho, \gamma)$ be a $\mathcal{C}$-decorated Kripke structure for some domain $\mathcal{C}$. We define an infinite path $\pi$ in $\mathbb{K}$ as a mapping $\pi : \mathbb{N} \to D$ such that $\pi(i) \to \pi(i+1)$ for all $i \geq 0$. For an infinite path $\pi$ in $\mathbb{K}$ we define the $\mathcal{C}$-decorated Kripke path $\mathbb{P}_\pi = (\mathbb{N}, \mathsf{S}, \rho', \gamma')$, where $\rho'(n) = \rho(\pi(n))$ and $\gamma'(n, r) = \gamma(\pi(n), r)$. Note that we may have $\pi(i) = \pi(j)$ for $i \neq j$. Given $d \in D$ and an $\mathsf{ECTL}^*(\tau)$-formula $\varphi$, we define $(\mathbb{K}, d) \models \varphi$ inductively as follows:

- $(\mathbb{K}, d) \models \varphi_1 \wedge \varphi_2$ iff $(\mathbb{K}, d) \models \varphi_1$ and $(\mathbb{K}, d) \models \varphi_2$.
- $(\mathbb{K}, d) \models \neg\varphi$ iff it is not the case that $(\mathbb{K}, d) \models \varphi$.
- $(\mathbb{K}, d) \models \mathsf{E}\psi(\varphi_1, \ldots, \varphi_m)$ iff there is an infinite path $\pi$ in $\mathbb{K}$ with $d = \pi(0)$ and $\mathbb{P}_\pi \models \psi(A_1, \ldots, A_m)$ where $A_i = \{j \in \mathbb{N} \mid (\mathbb{K}, \pi(j)) \models \varphi_i\}$.

The intuition behind the last point is that the sets $A_1, \ldots, A_m$ collect all the positions of the path $\mathbb{P}$ in which the formulas $\varphi_1, \ldots, \varphi_m$ hold. The free variables $X_1, \ldots, X_m$ from $\psi$ are then interpreted as $A_1, \ldots, A_m$ so that the formula $x \in X_i$ expresses that $\varphi_i$ holds at the $x$-th position of the path $\mathbb{P}$.

Note that for checking $(\mathbb{K}, d) \models \varphi$ we may ignore all propositions $p \in \mathsf{P}$ and all register variables $r \in \mathsf{Reg}$ that do not occur in $\varphi$.

Given a class of $\tau$-structures $\Gamma$, $\mathsf{SAT\text{-}ECTL}^*(\Gamma)$ denotes the following computational problem: *Given a formula* $\varphi \in \mathsf{ECTL}^*(\tau)$*, is there a concrete domain* $\mathcal{C} \in \Gamma$ *and a* $\mathcal{C}$*-decorated Kripke structure* $\mathbb{K} = (D, \rightarrow, \rho, \gamma)$ *such that* $(\mathbb{K}, d) \models \varphi$ *for some* $d \in D$*?* We also write $\mathsf{SAT\text{-}ECTL}^*(\mathcal{C})$ instead of $\mathsf{SAT\text{-}ECTL}^*(\{\mathcal{C}\})$.

## 2.4 Constraint $\mathsf{ECTL}^*$ and definable homomorphisms

Remember that we focus our interest in this paper on the satisfiability problem with respect to a class of structures over the signature $\tau = \{<, \perp, =\}$, where $=$ is always interpreted as equality, $<$ is interpreted as a partial order, and $\perp$ is interpreted as the incomparability relation with respect to $<$. In [8], we provided a connection between $\mathsf{SAT\text{-}ECTL}^*(\mathcal{A})$ for a $\tau$-structure $\mathcal{A}$ and the definability of the existence of a homomorphism to $\mathcal{A}$ in the logic $\mathsf{BMWB}$. To be more precise, we are interested in the definability of the existence of a homomorphism to the $\{<, \perp\}$-reduct of $\mathcal{A}$. In order to facilitate the presentation of this connection, we fix a class $\Gamma$ of $\{<, \perp\}$-structures.

For every structure $\mathcal{A} = (A, I) \in \Gamma$ we denote by $\mathcal{A}^=$ its expansion by equality, i.e., the $\tau$-structure $(A, J)$ where $J(<) = I(<)$, $J(\perp) = I(\perp)$, and $J(=) = \{(a, a) \mid a \in A\}$. Similarly, we set $\Gamma^= = \{\mathcal{A}^= \mid \mathcal{A} \in \Gamma\}$.

We call $\Gamma^=$ *negation-closed* if for every $R \in \{<, \perp, =\}$ there is a positive existential first-order formula $\varphi_R(x_1, \ldots, x_{\mathsf{ar}(R)})$ (i.e., a formula that is built up from atomic formulas using $\wedge$, $\vee$, and $\exists$) such that for all $\mathcal{A} = (A, I) \in \Gamma$

$$A^{\mathsf{ar}(R)} \setminus I(R) = \{(a_1, \ldots, a_{\mathsf{ar}(R)}) \mid \mathcal{A} \models \varphi_R(a_1, \ldots, a_{\mathsf{ar}(R)})\}.$$

In other words, the complement of every relation $I(R)$ must be definable by a positive existential first-order formula (which is independent from the specific structure $\mathcal{A}$).

*Example 5* For any class $\Delta$ of $\{<, \perp\}$-structures such that in every $\mathcal{A} \in \Delta$, (i) $<$ is interpreted as a strict partial order and (ii) $\perp$ is interpreted as the incomparability with respect to $<$ (i.e., $x \perp y$ iff neither $x \leq y$ nor $y \leq x$), $\Delta^=$ is negation-closed: For every $\mathcal{A} \in \Delta$ the following equalities hold in $\mathcal{A}^=$, where $A$ is the universe of $\mathcal{A}$:

$$(A^2 \setminus <) = \{(x, y) \mid \mathcal{A} \models y < x \vee y = x \vee x \perp y\}$$
$$(A^2 \setminus \perp) = \{(x, y) \mid \mathcal{A} \models x < y \vee x = y \vee y < x\}$$
$$(A^2 \setminus =) = \{(x, y) \mid \mathcal{A} \models x < y \vee x \perp y \vee y < x\}$$

In particular, the class of all semi-linear orders and all its subclasses are negation-closed. Note that for this it is crucial that we add the incomparability relation $\perp$.

**Definition 6** We say that $\Gamma$ has the property EHD (*existence of a homomorphism to a structure from $\Gamma$ is* BMWB-*definable*) if there is a BMWB-sentence $\varphi$ such that for every countable $\{<, \perp\}$-structure $\mathcal{B}$

$$\mathcal{B} \preceq \mathcal{A} \text{ for some } \mathcal{A} \in \Gamma \quad \Longleftrightarrow \quad \mathcal{B} \models \varphi.$$

Now the following theorem connects SAT-ECTL$^*(\Gamma^=)$ with EHD for the class $\Gamma$.

**Theorem 7 ([8])** *Let $\Gamma$ be a class of structures over $\{<, \perp\}$. If $\Gamma^=$ is negation-closed and $\Gamma$ has the EHD-property, then the problem* SAT-ECTL$^*(\Gamma^=)$ *is decidable.*

In the next sections, we show that the following classes of tree-like structures have the EHD-property:

1. the class of all semi-linear orders,
2. the class of all ordinal trees, and
3. for each $h \in \mathbb{N}$ the class of all trees of height $h$.

Thus, Theorem 7 shows that for these classes, the satisfiability problems for ECTL* with constraints are decidable, which proves our main Theorem 2.

Note that for a countable $\{<, \perp\}$-structure $\mathcal{B}$ there exists a homomorphism from $\mathcal{B}$ to a semi-linear order if and only if there exists a homomorphism from $\mathcal{B}$ to a countable semi-linear order, and analogous statements hold for ordinal trees and trees of height $h$. Hence, one can restrict the structure classes in Theorem 2 to its countable members.

## 3 Constraint ECTL* over semi-linear orders

Let $\Gamma$ denote the class of all semi-linear orders (over the signature $\{<, \perp\}$). The aim of this section is to prove that $\Gamma$ has the EHD-property. For this purpose, we characterize all those structures that admit a homomorphism to some element of $\Gamma$. The resulting criterion can be easily translated into WMSO. Hence, we do not need the bounding quantifier from WMSO+B here (the same will be true in the following Sections 4 and 5).

It turns out that, in the case of semi-linear orders (and also ordinal forests) the existence of such a homomorphism is in fact equivalent to the existence of a *compatible expansion*. We say that a graph[2] $(A, <, \perp)$ can be *extended* to a semi-linear order (an ordinal forest) if there is a partial order $\lhd$ such that $(A, \lhd, \perp_\lhd)$ is a semi-linear order (a ordinal forest) *compatible* with $(A, <, \perp)$, i.e.,

$$\forall x, y \in A \colon x < y \Rightarrow x \lhd y \text{ and } x \perp y \Rightarrow x \perp_\lhd y. \tag{2}$$

**Lemma 8** *The following are equivalent for every structure $\mathcal{A} = (A, <, \perp)$:*

1. *$\mathcal{A}$ can be extended to a semi-linear order (to an ordinal forest, resp.).*

[2] We call $(A, <, \perp)$ a graph to emphasize that here the binary relation symbols $<$ and $\perp$ can have arbitrary interpretations and they need not be a partial order and its incomparability relation. We can instead see them as two different kinds of edges in an arbitrary graph.

*2.* $\mathcal{A} \preceq \mathcal{B}$ *for some semi-linear order (ordinal tree, resp.)* $\mathcal{B}$*.*

*Proof* We start with the implication $(1 \Rightarrow 2)$. Assume that $\mathcal{A}$ can be extended to a compatible semi-linear order (ordinal forest, resp.) $\mathcal{A}' = (A, \lhd, \perp_\lhd)$. Thanks to compatibility, the identity is a homomorphism from $\mathcal{A}$ to $\mathcal{A}'$. In the case of an ordinal forest, one can add one common minimal element to obtain an ordinal tree.

Let us now prove the implication $(2 \Rightarrow 1)$. Suppose that $h$ is a homomorphism from $\mathcal{A} = (A, <, \perp)$ to some semi-linear order $\mathcal{B} = (B, \prec, \perp_\prec)$. We extend $\mathcal{A}$ to a compatible semi-linear order $(A, \lhd, \perp_\lhd)$. Let us fix an arbitrary well-order $<_{\mathrm{wo}}$ on the set $A$ (which exists by the axiom of choice). We define the binary relation $\lhd$ on $A$ as follows:

$$x \lhd y \text{ if and only if } h(x) \prec h(y) \text{ or } (h(x) = h(y) \text{ and } x <_{\mathrm{wo}} y).$$

As usual, we denote with $\perp_\lhd$ the incomparability relation for $\lhd$, i.e., $x \perp_\lhd y$ if and only if neither $x \lhd y$ nor $y \lhd x$ nor $x = y$ holds. We show that $(A, \lhd, \perp_\lhd)$ is a semi-linear order. In fact, irreflexivity and transitivity of $\lhd$ are easy consequences of the definition of $\lhd$ and of the fact that $\prec$ is a partial order. To show that $\lhd$ is semi-linear, assume that $x_1 \lhd x$ and $x_2 \lhd x$. By definition $h(x_1) \prec h(x)$ or $h(x_1) = h(x)$ and $h(x_2) \prec h(x)$ or $h(x_2) = h(x)$. By semi-linearity of $\mathcal{B}$, we deduce that $h(x_1)$ and $h(x_2)$ are comparable and, by definition of $\lhd$, so are $x_1$ and $x_2$. It remains to show that $(A, \lhd, \perp_\lhd)$ is compatible with $\mathcal{A}$. Let $x < y$. Then, by the fact that $h$ is a homomorphism, $h(x) \prec h(y)$ which guarantees that $x \lhd y$. If $x \perp y$, then $h(x) \perp_\prec h(y)$. Since $\mathcal{B}$ is a semi-linear order, this implies that neither $h(x) \prec h(y)$ nor $h(y) \prec h(x)$ nor $h(x) = h(y)$ holds. As a consequence none of $x \lhd y$, $y \lhd x$ and $x = y$ holds. Therefore we have $x \perp_\lhd y$.

The case in which $\mathcal{B}$ is an ordinal tree is dealt with similarly. It is enough to notice that $\lhd$ does not contain any infinite decreasing chains, since $\prec$ is well-founded and $<_{\mathrm{wo}}$ is a well-order. □

Inspired by Wolk's work on comparability graphs [21,22] we use Rado's selection lemma [17] in order to obtain the compactness result that a graph can be extended to a semi-linear order iff every finite subgraph can be extended to a semi-linear order. Recall that a choice function for a family of sets $X = \{X_i \mid i \in I\}$ is a function $f$ with domain $I$ such that $f(i) \in X_i$ for all $i \in I$, i.e., it *chooses* one element from each set $X_i$.

**Lemma 9 (Rado's selection lemma, cf. [16,17])** *Let* $I$ *be an arbitrary index set and let* $X = \{X_i \mid i \in I\}$ *be a family of finite sets. For each finite subset* $A$ *of* $I$*, let* $f_A$ *be a choice function for the family* $\{X_i \mid i \in A\}$*. Then there is a choice function* $f$ *for* $X$ *such that, for all finite* $A \subseteq I$*, there is a finite set* $B$ *such that* $A \subseteq B \subseteq I$ *with* $f(i) = f_B(i)$ *for all* $i \in A$*.*

**Lemma 10 (extension of Theorem 2 in [22])** *A structure* $\mathcal{A} = (A, <, \perp)$ *can be extended to a semi-linear order if and only if every finite substructure of* $\mathcal{A}$ *can be extended to a semi-linear order.*

*Proof* The direction $(\Rightarrow)$ is trivial. For the direction $(\Leftarrow)$ let

$$I = \{\{x, y\} \mid x, y \in A, x \neq y\}$$

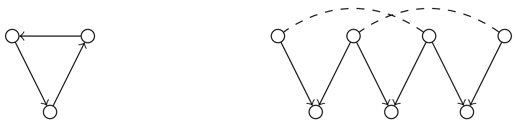

**Fig. 1** A <-cycle of three elements and an "incomparable triple-u", where dashed lines are $\perp$-edges.

be the set of 2-elementary subsets of $A$. For all $i = \{x, y\} \in I$ we define $Z_i = \{(x, y), (y, x), \#\}$. We want to find a choice function for the family of sets $\{Z_i \mid i \in I\}$ which is in some sense *compatible* with the relations $\perp$ and $<$. In fact, choosing for each $i \in I$ one element of $Z_i$ corresponds intuitively to deciding whether the two elements $x$ and $y$ are comparable, and in which order, or if they are incomparable.

Each finite subset $J$ of $I$ defines a set $\bar{J} = \{x \in j \mid j \in J\} \subseteq A$ and a substructure $\mathcal{A}_{\restriction \bar{J}} = (\bar{J}, <_{\restriction \bar{J}}, \perp_{\restriction \bar{J}})$. Since $\mathcal{A}_{\restriction \bar{J}}$ is finite, by hypothesis it can be extended to a semi-linear order. Hence, we can find a partial order $\lhd_J$ on $\bar{J}$ such that $(\bar{J}, \lhd_J, \perp_{\lhd_J})$ is a semi-linear order compatible with $\mathcal{A}_{\restriction \bar{J}}$ as in (2) on page 9.

Let $f_J$ be the following choice function for $\{Z_j \mid j \in J\}$, where $\{x, y\} \in J$:

$$f_J(\{x, y\}) = \begin{cases} (y, x) & \text{iff } y \lhd_J x, \\ (x, y) & \text{iff } x \lhd_J y, \\ \# & \text{otherwise.} \end{cases}$$

By Lemma 9 we can find a choice function $f$ for $\{Z_i \mid i \in I\}$ such that for all finite $J \subseteq I$ there is a finite set $K$ such that

$$J \subseteq K \subseteq I \text{ and } f(j) = f_K(j) \text{ for all } j \in J.$$

Define $x \lhd y$ iff $(x, y) = f(\{x, y\})$. We need to prove that $(A, \lhd, \perp_{\lhd})$ is an extension of $\mathcal{A}$ to a semi-linear order. But all the properties that we need to check are local, and on every finite subset of $A$, $\lhd$ coincides with some $\lhd_J$, which is a semi-linear order compatible with $<$ and $\perp$. ☐

Thanks to Lemma 10, given a $\{<, \perp\}$-structure $\mathcal{A}$, proving EHD only requires to look for a necessary and sufficient condition which guarantees that every finite substructure of $\mathcal{A}$ can be extended to a semi-linear order.

**Definition 11** Let $\mathcal{A} = (A, <, \perp)$ be a graph. Given $A' \subseteq A$, we say that $A'$ is *connected (with respect to <)* if and only if, for all $a, a' \in A'$ , there are $a_1, \dots, a_n \in A'$ such that $a = a_1$, $a' = a_n$ and $a_i < a_{i+1}$ or $a_{i+1} < a_i$ for all $1 \leq i \leq n-1$. A *connected component* of $\mathcal{A}$ is a maximal (with respect to inclusion) connected subset of $A$. Given a subset $A' \subseteq A$ and $c \in A'$, we say that $c$ is a *central point of* $A'$ if and only if for every $a \in A'$ neither $a \perp c$ nor $c \perp a$ nor $a < c$ holds.

In other words, a central point of a subset $A' \subseteq A$ is a node of the structure $\mathcal{A} = (A, <, \perp)$ which has no incoming or outgoing $\perp$-edges, and no incoming $<$-edges within $A'$.

*Example 12* A <-cycle (of any number of elements) does not have a central point, nor does an *incomparable triple-u*, see Figure 1. Both structures do not admit a homomorphism into a semi-linear order. While this statement is obvious for the cycle, we leave the proof for the incomparable triple-u as an exercise.

**Lemma 13** *A finite structure $\mathcal{A} = (A, <, \perp)$ can be extended to a semi-linear order if and only if every non-empty connected $B \subseteq A$ has a central point.*

Let us extract the main argument for the ($\Rightarrow$)-part of the proof for later reuse:

**Lemma 14** *Let $(A, \lhd, \perp_\lhd)$ (with $A$ not necessarily finite) be a semi-linear order extending $\mathcal{A} = (A, <, \perp)$. If a connected subset $B \subseteq A$ (with respect to $<$) contains a minimal element $m$ with respect to $\lhd$, then $m$ is central in $B$ (again with respect to $\mathcal{A}$).*

*Proof* Let $b \in B$. Since $B$ is connected, there are $b_1, \dots, b_n \in B$ such that $b_1 = m$, $b_n = b$ and $b_i < b_{i+1}$ or $b_{i+1} < b_i$ for all $1 \le i \le n-1$. As $\lhd$ is compatible with $<$, this implies that $b_i \lhd b_{i+1}$ or $b_{i+1} \lhd b_i$ for all $1 \le i \le n-1$. Given that $m$ is minimal, applying semi-linearity of $\lhd$, we obtain that $m = b_i$ or $m \lhd b_i$ for all $1 \le i \le n$. In particular, we have $m = b$ or $m \lhd b$. Since $(A, \lhd, \perp_\lhd)$ is a semi-linear order compatible with $(A, <, \perp)$, we cannot have $b < m$, $m \perp b$ or $b \perp m$ (since this would imply $b \lhd m$ or $m \perp_\lhd b$). Hence, $m$ is central. ☐

*Proof (of Lemma 13)* For the direction ($\Rightarrow$) assume that $\mathcal{A}$ can be extended to a semi-linear order and let $B$ be any non-empty connected subset of $A$. Since $B$ is finite, there is a minimal element $m$ (with respect to the semi-linear order extending $\mathcal{A}$). Using Lemma 14 we conclude that $m$ is central in $B$.

We prove the direction ($\Leftarrow$) by induction on $n = |A|$. Suppose $n = 1$ and let $A = \{a\}$. The fact that $\{a\}$ has a central point implies that neither $a < a$ nor $a \perp a$ holds. Hence, $\mathcal{A}$ is a semi-linear order.

Suppose now that $n > 1$ and assume the statement to be true for all $i < n$. If $\mathcal{A}$ is not connected with respect to $<$, then we apply the induction hypothesis to every connected component. The union of the resulting semi-linear orders extends $\mathcal{A}$. Now assume that $\mathcal{A}$ is connected and let $c$ be a central point of $A$. By the induction hypothesis we can find $\lhd'$ such that $(A \setminus \{c\}, \lhd', \perp_{\lhd'})$ is a semi-linear order extending $\mathcal{A} \setminus \{c\}$. We define $\lhd := \lhd' \cup \{(c, x) \mid x \in A \setminus \{c\}\}$ (i.e., we add $c$ as a smallest element), which is obviously a partial order on $A$.

To prove that $\lhd$ is semi-linear, let $a_1, a_2, a \in A$ such that $a_1 \lhd a$ and $a_2 \lhd a$. If $a_1 = c$ or $a_2 = c$, then $a_1$ and $a_2$ are comparable by definition. Otherwise, we conclude that $a_1, a_2, a \in A \setminus \{c\}$. Hence, $a_1 \lhd' a$ and $a_2 \lhd' a$, and semi-linearity of $\lhd'$ settles the claim.

We finally show compatibility. Suppose that $a < b$. If $a = c$, then $a \lhd b$. The case $b = c$ cannot occur, because $c$ is central in $A$. The remaining possibility $a \neq c \neq b$ implies that $a \lhd' b$ and hence $a \lhd b$ as desired. Finally, suppose that $a \perp b$. Then $a \neq c \neq b$, because $c$ is central. We conclude that $a \perp_{\lhd'} b$ and also $a \perp_\lhd b$. ☐

We are finally ready to state the main result of this section which (together with Theorem 7) completes the proof of statement (1) of Theorem 2:

**Proposition 15** *The class of all semi-linear orders $\Gamma$ has the* EHD*-property.*

*Proof* Take $\mathcal{A} = (A, <, \perp)$. Thanks to Lemmas 8, 10 and 13, it is enough to show that WMSO can express the condition that every finite and non-empty connected substructure of $\mathcal{A}$ has a central point. We define the following WMSO-formula $\mathsf{reach}(x, y, X)$ such that $\mathcal{A} \models \mathsf{reach}(a, b, B)$ if and only if $a$ and $b$ are in the same connected component of $\mathcal{A}{\restriction_B}$:

$$x \in X \wedge \forall Y \subseteq X \left[ \big( x \in Y \wedge \forall z \in Y \ \forall w \in X \, (\varphi(z, w) \to w \in Y) \big) \to y \in Y \right],$$

where $\varphi(z,w) := z < w \vee w < z$. Then, we define the following $\mathsf{WMSO}$-formulas:

$$\begin{aligned}\mathsf{connected}(X) &:= \forall x \in X\ \forall y \in X\ \mathsf{reach}(x,y,X),\\ \mathsf{central}(x,X) &:= x \in X \wedge \forall y \in X\ \neg(x \perp y \vee y \perp x \vee y < x), \text{ and}\\ \psi &:= \forall X\,(\mathsf{connected}(X) \wedge X \neq \emptyset \to \exists x\,\mathsf{central}(x,X)).\end{aligned}$$

It is straightforward to verify that $\mathcal{A} \models \psi$ if and only if every finite non-empty connected subset of $A$ has a central point. ◻

## 4 Constraint $\mathsf{ECTL}^*$ over ordinal trees

Let $\Omega$ denote the class of all ordinal trees (over the signature $\{<,\perp\}$). The aim of this section is to prove that $\Omega$ has the $\mathsf{EHD}$-property. We use again the notions of a connected subset and a central point as introduced in Definition 11. We will characterize those structures which admit a homomorphism into an ordinal tree. Here, in contrast with the case of semi-linear orders, the final condition will be that all connected sets (not just the finite ones) have a central point.

**Lemma 16** *Let $\mathcal{A} = (A,<,\perp)$ be a structure. There exists $\mathcal{O} \in \Omega$ such that $\mathcal{A} \preceq \mathcal{O}$ if and only if every non-empty and connected $B \subseteq A$ has a central point.*

*Proof* We start with the direction ($\Rightarrow$). Due to Lemma 8 we can assume that there is a relation $\lhd$ that extends $(A,<,\perp)$ to an ordinal forest. Let $B \subseteq A$ be a non-empty connected set. Since $(A,\lhd,\perp_\lhd)$ is an ordinal forest, $B$ has a minimal element $c$ with respect to $\lhd$. By Lemma 14, $c$ is a central point of $B$.

For the direction ($\Leftarrow$) we first define a partition of the domain of $\mathcal{A}$ into subsets $C_\beta$ for $\beta \sqsubset \chi$, where $\chi$ is an ordinal (whose cardinality is bounded by the cardinality of $A$). Here $\sqsubset$ denotes the natural order on ordinals. Assume that the pairwise disjoint subsets $C_\beta$ have been defined for all $\beta \sqsubset \alpha$ (which is true for $\alpha = 0$ in the beginning). Then we define $C_\alpha$ as follows. Let $C_{\sqsubset\alpha} = \bigcup_{\beta\sqsubset\alpha} C_\beta \subseteq A$. If $A \setminus C_{\sqsubset\alpha}$ is not empty, then we define $\mathcal{C}_\alpha$ as the set of connected components of $A \setminus C_{\sqsubset\alpha}$. Let

$$C_\alpha = \{c \in A \setminus C_{\sqsubset\alpha} \mid c \text{ is a central point of some } B \in \mathcal{C}_\alpha\}.$$

Clearly, $C_\alpha$ is not empty. Hence, there must exist a smallest ordinal $\chi$ such that $A = C_{\sqsubset\chi}$.

For every ordinal $\alpha \sqsubset \chi$ and each element $c \in C_\alpha$ we define the sequence of connected components $\mathsf{road}(c) = (B_\beta)_{(\beta\sqsubseteq\alpha)}$, where $B_\beta \in \mathcal{C}_\beta$ is the unique connected component with $c \in B_\beta$. This ordinal-indexed sequence keeps record of the *road* we took to reach $c$ by storing information about the connected components to which $c$ belongs at each stage of our process.

Given $\mathsf{road}(c) = (B_\beta)_{(\beta\sqsubseteq\alpha)}$ and $\mathsf{road}(c') = (B'_\beta)_{(\beta\sqsubseteq\alpha')}$ for some $c \in C_\alpha$ and $c' \in C_{\alpha'}$, let us define $\mathsf{road}(c) \lhd \mathsf{road}(c')$ if and only if $\alpha \sqsubset \alpha'$ and $B_\beta = B'_\beta$ for all $\beta \sqsubseteq \alpha$. Basically this is the *prefix order* for ordinal-sized sequences of connected components.

Now let $O = \{\mathsf{road}(c) \mid c \in A\}$. Note that $\mathcal{O} = (O,\lhd,\perp_\lhd)$ is an ordinal forest, because for each $c \in C_\alpha$ the order $(\{\mathsf{road}(c') \mid \mathsf{road}(c') \unlhd \mathsf{road}(c)\},\unlhd)$ forms the ordinal $\alpha$ (for each $\beta \sqsubset \alpha$ it contains exactly one road of length $\beta$).

Now we show that the mapping $c \mapsto \mathsf{road}(c)$ $(c \in A)$ is a homomorphism from $\mathcal{A}$ to $\mathcal{O}$. Take elements $a, a' \in A$ with $a \in C_\alpha$, and $a' \in C_{\alpha'}$ for some $\alpha,\alpha' \sqsubset \chi$. Let $\mathsf{road}(a) = (B_\beta)_{(\beta\sqsubseteq\alpha)}$ and $\mathsf{road}(a') = (B'_\beta)_{(\beta\sqsubseteq\alpha')}$.

– If $a < a'$, then (i) $\alpha \sqsubset \alpha'$, because $a'$ cannot be central point of a set which contains $a$, and (ii) $B_\beta = B'_\beta$ for all $\beta \sqsubseteq \alpha$ because $a$ and $a'$ belong to the same connected component of $A \setminus C_{\sqsubset\beta}$ for all $\beta \sqsubseteq \alpha$. By these observations we deduce that $\mathsf{road}(a) \lhd \mathsf{road}(a')$.
– If $a \perp a'$, then, without loss of generality, suppose that $\alpha \sqsubseteq \alpha'$. At stage $\alpha$, $a$ is a central point of $B_\alpha \in \mathcal{C}_\alpha$. Since $\alpha \sqsubseteq \alpha'$, the connected component $B'_\alpha$ exists. We must have $B_\alpha \neq B'_\alpha$, since otherwise we would have $a \perp a' \in B_\alpha$ contradicting the fact that $a$ is central for $B_\alpha$. Therefore, $\mathsf{road}(a) \perp_\lhd \mathsf{road}(a')$.

We finally add one extra element $\mathsf{road}_0$ and make this the minimal element of $\mathcal{O}$, thus finding a homomorphism from $\mathcal{A}$ into an ordinal tree. ◻

We can now complete the proof of statement (2) from Theorem 2.

**Proposition 17** *The class $\Omega$ of all ordinal trees has the EHD-property.*

*Proof* Given a $\{<, \perp\}$-structure $\mathcal{A}$, it suffices by Lemma 16 to find an MSO-formula expressing the fact that every non-empty connected subset of $\mathcal{A}$ has a central point. Recall the WMSO-formula $\psi$ from the proof of Theorem 15. Seen as an MSO-formula, $\psi$ clearly does the job. ◻

*Remark 18* The procedure described in the proof of Lemma 16 can be also used to embed a structure $\mathcal{A} = (A, <, \perp)$ into an ordinary tree (where for every $x$, the set of all elements smaller than $x$ forms a finite linear order). For this, the ordinal $\chi$ has to satisfy $\chi \leq \omega$, i.e., every element $a \in A$ has to belong to a set $C_n$ for some finite $n$. We use this observation in Section 6. Unfortunately, our results from Section 6 imply that we cannot express in BMWB that the above procedure terminates at stage $\omega$.

## 5 Constraint ECTL* over trees of height $h$

Fix $h \in \mathbb{N}$. The aim of this section is to show that the class $\Theta_h$ of all trees of height $h$ (over $\{<, \perp\}$) has the EHD-property. The proof relies on the fact that we can unfold the fixpoint procedure on the central points from the ordinal tree setting for $h$ steps in MSO.

For this section, we fix an arbitrary structure $\mathcal{A} = (A, <, \perp)$. We first define subsets $A_0, A_1, \dots, A_h \subseteq A$ that are pairwise disjoint. The elements of $A_0$ are the central points of $A$ (this set is possibly empty) and, for each $i \geq 1$, $A_i$ contains the central points of each connected component of $A \setminus (A_0 \cup \dots \cup A_{i-1})$. Note that $A_0$ contains exactly those nodes of $\mathcal{A}$ that a homomorphism from $\mathcal{A}$ to some tree can map to the root of the tree because elements from $A_0$ are neither incomparable to any other element nor larger than any other element, while all element outside of $A_0$ have to be incomparable to some other element or have to be larger than some other element. Hence they cannot be mapped to the root by any homomorphism. Thus, there is a homomorphism from $\mathcal{A}$ to some element of $\Theta_h$ if and only if $\mathcal{A} \setminus A_0$ can be embedded into some forest of height $h-1$. Now the sets $A_i$ for $1 \leq i \leq h$ collect exactly those elements which are chosen in the $i$-th step of the fixpoint procedure from the proof of Lemma 16 (where this set is called $C_i$). Thus, if $A_0, A_1, \dots, A_h$ form a partition of $A$, then $\mathcal{A}$ allows a homomorphism to some $\mathcal{T} \in \Theta_h$. It turns out that the converse is also true. If $\mathcal{A} \preceq \mathcal{T}$ for some $\mathcal{T} \in \Theta_h$ then $A_0, A_1, \dots, A_h$ form a partition of $A$. Thus, it suffices to show that each $A_i$ is MSO-definable. To do this, we define for all $i \in \mathbb{N}$ the formulas

$$\begin{aligned} \varphi_0(x) &:= \forall y \, \neg(y < x \vee y \perp x \vee x \perp y), & (3)\\ \mathsf{rest}_{i+1}(x) &:= \bigwedge_{j \le i} \neg\varphi_j(x), \\ \mathsf{con}_{i+1}(x,y) &:= \exists Z \, \forall z \, (z \in Z \to \mathsf{rest}_{i+1}(z)) \wedge \mathsf{reach}(x,y,Z), \\ \varphi_{i+1}(x) &:= \mathsf{rest}_{i+1}(x) \wedge \forall y \, (\mathsf{con}_{i+1}(x,y) \to \neg(y < x \vee y \perp x \vee x \perp y)), & (4) \end{aligned}$$

where $\mathsf{reach}(x,z,Z)$ is defined as in the proof of Proposition 15 on page 12. For $i \ge 0$, let $A_i$ be the set of nodes $a \in A$ such that $\mathcal{A} \models \varphi_i(a)$.

Clearly $A_0$ is the set of central points of $A$. Inductively, one shows that $A_{i+1}$ is the set of central points of the connected components of $A \setminus (A_0 \cup \dots \cup A_i)$.

**Lemma 19** *There exists $\mathcal{T} \in \Theta_h$ such that $\mathcal{A} \preceq \mathcal{T}$ if and only if $A_0, A_1, \dots, A_h$ is a partition of $A$.*

*Proof* For the implication ($\Rightarrow$) take a homomorphism $g$ from $\mathcal{A}$ to a tree $\mathcal{T} = (T, \lhd, \perp_\lhd) \in \Theta_h$. By induction we prove that if $g$ maps $a$ to the $i$-th level of $\mathcal{T}$ then $a \in A_j$ for some $j \le i$. For $i = 0$ assume that $g(a)$ is the root of the tree. Then $a$ cannot be incomparable or greater than any other element. Thus, it is a central point of $A$, i.e., $a \in A_0$.

For the inductive step, assume that $g(a)$ is on the $i$-th level for $i > 0$. Heading for a contradiction, assume that $a$ is neither in $A_0 \cup \dots \cup A_{i-1}$ nor a central point of some connected component of $A \setminus (A_0 \cup \dots \cup A_{i-1})$. Then there is some $a' \in A \setminus (A_0 \cup \dots \cup A_{i-1})$ such that $a$ and $a'$ are in the same connected component of $A \setminus (A_0 \cup \dots \cup A_{i-1})$ and one of $a' < a$, $a' \perp a$ or $a \perp a'$ holds. Since $g$ is a homomorphism, we get $g(a') \lhd g(a)$ or $g(a') \perp_\lhd g(a)$. If $g(a') \lhd g(a)$, then $a'$ has to be mapped by $g$ to some level $j < i$, whence $a' \in A_0 \cup \dots \cup A_j$ by the induction hypothesis. This contradicts our assumption on $a'$. Now, assume that $g(a') \perp_\lhd g(a)$. Let $a = a_0, a_1, \dots, a_m = a'$ be a $<$-path connecting $a$ and $a'$ in $A \setminus (A_0 \cup \dots \cup A_{i-1})$. Since $a_i \notin A_0 \cup \dots \cup A_{i-1}$, the induction hypothesis shows that all $g(a_i)$ are on level $i$ or larger. But then, since $a_0, a_1, \dots, a_m$ is a path, all $g(a_i)$ must belong to the subtree rooted at $g(a)$. In particular, $g(a') = g(a_m)$ is comparable to $g(a)$, which contradicts $g(a') \perp_\lhd g(a)$. Thus, we can conclude that $a \in A_0 \cup \dots \cup A_i$.

For the direction ($\Leftarrow$) assume that $A_0 \cup \dots \cup A_h = A$. Applying the same construction described in the proof of Lemma 16 for ordinal trees, it is not hard to see that we find a homomorphism $g$ from $\mathcal{A}$ to some tree of height $h$ which maps the elements of $A_i$ to elements on level $i$. Should $A_0$ be empty, then $A$ would not be connected, and we would have a forest of height $h-1$. Adding a minimal element we still get a tree of height $h$. ◻

**Theorem 20** *$\Theta_h$ has the EHD-property.*

*Proof* Let $\mathcal{A}$ be any $\{<, \perp\}$-structure. Then, by Lemma 19, $\mathcal{A} \preceq \mathcal{T}$ for some $\mathcal{T} \in \Theta_h$ if and only if

$$\mathcal{A} \models \forall x \bigvee_{i=0}^{h} \varphi_i(x),$$

where the formulas $\varphi_i$ are defined in (3) and (4). ◻

## 6 Trees do not have the EHD-property

Let $\Theta$ be the class of all countable trees (over $\{<, \perp\}$). In this section, we prove that the logic BMWB (the most expressive logic for which the EHD-method currently works) cannot distinguish between graphs that admit a homomorphism to some element of $\Theta$ and those that do not. Heading for a contradiction, assume that $\varphi$ is a sentence such that a countable structure $\mathcal{A} = (A, <, \perp)$ satisfies $\varphi$ if and only if there is a homomorphism from $\mathcal{A}$ to some $\mathcal{T} \in \Theta$. Let $k$ be the quantifier rank of $\varphi$. We construct two graphs $\mathcal{E}_k$ and $\mathcal{U}_k$ such that $\mathcal{E}_k$ admits a homomorphism into a tree while $\mathcal{U}_k$ does not. We then use an Ehrenfeucht-Fraïssé game for BMWB to show that $\varphi$ cannot separate these two structures, contradicting our assumption. This contradiction shows that $\Theta$ does not have the EHD-property, proving our second main result Theorem 1.

### 6.1 The WMSO+B-Ehrenfeucht-Fraïssé-game

The $k$-round WMSO+B-EF-game on a pair of structures $(\mathcal{A}, \mathcal{B})$ over the same finite relational signature $\sigma$ is played by spoiler and duplicator as follows.[3] In the following, $A$ denotes the domain of $\mathcal{A}$ and $B$ the domain of $\mathcal{B}$.

The game starts in position

$$p_0 := (\mathcal{A}, \emptyset, \emptyset, \mathcal{B}, \emptyset, \emptyset).$$

In general, before playing the $i$-th round (for $1 \leq i \leq k$) the game is in a position

$$p = (\mathcal{A}, a_1, \dots, a_{i_1}, A_1, \dots, A_{i_2}, \mathcal{B}, b_1, \dots, b_{i_1}, B_1, \dots, B_{i_2}),$$

where

1. $i_1, i_2 \in \mathbb{N}$ satisfy $i_1 + i_2 = i - 1$,
2. $a_j \in A$ for all $1 \leq j \leq i_1$,
3. $b_j \in B$ for all $1 \leq j \leq i_1$,
4. $A_j \subseteq A$ is a finite set for all $1 \leq j \leq i_2$, and
5. $B_j \subseteq B$ is a finite set for all $1 \leq j \leq i_2$.

In the $i$-th round spoiler and duplicator produce the next position as follows. Spoiler chooses to play one of the following three possibilities:

1. Spoiler can play an *element move*. For this he chooses either some $a_{i_1+1} \in A$ or $b_{i_1+1} \in B$. Duplicator then responds with an element from the other structure, i.e., with $b_{i_1+1} \in B$ or $a_{i_1+1} \in A$. The position in the next round is
$$(\mathcal{A}, a_1, \dots, a_{i_1+1}, A_1, \dots, A_{i_2}, \mathcal{B}, b_1, \dots, b_{i_1+1}, B_1, \dots, B_{i_2}).$$
2. Spoiler can play a *set move*. For this he chooses either some finite $A_{i_2+1} \subseteq A$ or some finite $B_{i_2+1} \subseteq B$. Duplicator then responds with a finite set from the other structure, i.e., with $B_{i_2+1} \subseteq B$ or $A_{i_2+1} \subseteq A$. The position in the next round is
$$(\mathcal{A}, a_1, \dots, a_{i_1}, A_1, \dots, A_{i_2+1}, \mathcal{B}, b_1, \dots, b_{i_1}, B_1, \dots, B_{i_2+1}).$$

[3] For the ease of presentation we assume that $\mathcal{A}$ and $\mathcal{B}$ are infinite structures.

3. Spoiler can play a *bound move.* For this he chooses one of the structures $\mathcal{A}$ or $\mathcal{B}$ and chooses a natural number $l \in \mathbb{N}$. Duplicator responds with another number $m \in \mathbb{N}$. Then the game continues as in the case of a set move with the restrictions that spoiler has to choose a finite subset of size at least $m$ from his chosen structure and duplicator has to respond with a finite subset of size at least $l$ from the other structure.

After $k$ rounds, the game ends in a position

$$p = (\mathcal{A}, a_1, \ldots, a_{i_1}, A_1, \ldots, A_{i_2}, \mathcal{B}, b_1, \ldots, b_{i_1}, B_1, \ldots, B_{i_2}).$$

Duplicator wins the game if

1. $a_j \in A_k \Leftrightarrow b_j \in B_k$ for all $1 \leq j \leq i_1$ and all $1 \leq k \leq i_2$,
2. $a_j = a_k \Leftrightarrow b_j = b_k$ for all $1 \leq j < k \leq i_1$, and
3. for all relation symbols $R \in \sigma$ (of arity $n$)

$$(a_{j_1}, \ldots, a_{j_n}) \in R^{\mathcal{A}} \Leftrightarrow (b_{j_1}, \ldots, b_{j_n}) \in R^{\mathcal{B}}$$

for all $j_1, \ldots, j_n \in \{1, \ldots, i_1\}$.

As one would expect, the $\mathsf{WMSO{+}B}$-EF-game can be used to show undefinability results for $\mathsf{WMSO{+}B}$ due to the relationship between winning strategies in the $k$-round game and equivalence with respect to formulas up to quantifier rank $k$.

**Proposition 21** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\sigma$-structures. Given the elements $a_1, \ldots, a_{i_1} \in \mathcal{A}$, $b_1, \ldots, b_{i_1} \in \mathcal{B}$, and the finite sets $A_1, \ldots, A_{i_2} \subseteq \mathcal{A}$, $B_1, \ldots, B_{i_2} \subseteq \mathcal{B}$, we define the position*

$$p = (\mathcal{A}, a_1, \ldots, a_{i_1}, A_1, \ldots, A_{i_2}, \mathcal{B}, b_1, \ldots, b_{i_1}, B_1, \ldots, B_{i_2}).$$

*Then, $(\mathcal{A}, a_1, \ldots, a_{i_1}, A_1, \ldots, A_{i_2})$ and $(\mathcal{B}, b_1, \ldots, b_{i_1}, B_1, \ldots, B_{i_2})$ are indistinguishable by any $\mathsf{WMSO{+}B}$-formula $\varphi(x_1, \ldots, x_{i_1}, X_1, \ldots, X_{i_2})$ of quantifier rank $k$ if and only if duplicator has a winning strategy in the $k$-round $\mathsf{WMSO{+}B}$-EF-game started in position $p$.*

*Proof* First of all note that up to logical equivalence there are only finitely many different $\mathsf{WMSO{+}B}$-formulas $\varphi(x_1, \ldots, x_{i_1}, X_1, \ldots, X_{i_2})$ of quantifier rank $k$. This fact is proved in a completely analogous way to the case of first-order or monadic second-order logic.

The proof is by induction on $k$. The base case $k = 0$ is trivial. Assume now that the proposition holds for $k-1$. We use the abbreviations $\bar{a} = (a_1, \ldots, a_{i_1})$, $\bar{A} = (A_1, \ldots, A_{i_2})$, $\bar{b} = (b_1, \ldots, b_{i_1})$, and $\bar{B} = (B_1, \ldots, B_{i_2})$ in the following. First assume that there is a $\mathsf{WMSO{+}B}$-formula $\varphi(x_1, \ldots, x_{i_1}, X_1, \ldots, X_{i_2})$ of quantifier rank $k$ such that

$$\mathcal{A} \models \varphi(\bar{a}, \bar{A}) \tag{5}$$

and

$$\mathcal{B} \not\models \varphi(\bar{b}, \bar{B}). \tag{6}$$

We show that spoiler has a winning strategy in the $k$-round game by a case distinction on the structure of $\varphi$. We only consider the case $\varphi = \mathsf{B}X\psi$ (all other cases can be handled exactly as in the $\mathsf{WMSO}$-EF-game, see e.g. [14]). Let $l \in \mathbb{N}$ be a strict bound witnessing (5), in the sense that there is no set $A_{i_2+1}$ of size at least $l$ such that $\mathcal{A} \models \varphi(\bar{a}, \bar{A}, A_{i_2+1})$. Then spoiler chooses structure $\mathcal{B}$ and bound $l$. Duplicator responds with some bound $m \in \mathbb{N}$. Due to (6)

$$\mathcal{B} \models \neg \mathsf{B} X \psi(\bar{b}, \bar{B}, X).$$

Hence, there is a set $B_{i_2+1}$ of size at least $m$ such that

$$\mathcal{B} \models \psi(\bar{b}, \bar{B}, B_{i_2+1}).$$

Spoiler chooses this set $B_{i_2+1}$. Duplicator must answer with a set $A_{i_2+1}$ of size at least $l$. By the choice of $l$ we conclude that

$$\mathcal{A} \not\models \psi(\bar{a}, \bar{A}, A_{i_2+1}).$$

By the induction hypothesis, spoiler has a winning strategy in the resulting position for the $(k-1)$-round game.

For the other direction, assume that $(\mathcal{A}, \bar{a}, \bar{A})$ and $(\mathcal{B}, \bar{b}, \bar{B})$ are indistinguishable by WMSO+B-formulas of quantifier rank $k$. Duplicator's strategy is as follows:

- If spoiler plays an element move choosing without loss of generality $a_{i_1+1} \in \mathcal{A}$, let $\Phi$ be the set of all WMSO+B-formulas $\varphi$ of quantifier rank $k-1$ such that $\mathcal{A} \models \varphi(\bar{a}, a_{i_1+1}, \bar{A})$. Since $\Phi$ is finite up to logical equivalence, there is a WMSO+B-formula $\psi$ of quantifier rank $k-1$ such that $\psi \equiv \bigwedge_{\varphi \in \Phi} \varphi$. By the assumption (indistinguishability up to quantifier rank $k$) and the fact that $\mathcal{A} \models \exists x\, \psi(\bar{a}, x, \bar{A})$ we conclude that $\mathcal{B} \models \exists x\, \psi(\bar{b}, x, \bar{B})$. Hence, there is an element $b_{i_1+1} \in \mathcal{B}$ such that $\mathcal{B} \models \psi(\bar{b}, b_{i_1+1}, \bar{B})$. Thus, duplicator can respond with $b_{i_1+1}$ and obtain a position for which he has a winning strategy by the induction hypothesis.
- If spoiler plays a set move, we use the same strategy as in the element move. We only have to replace the element $a_{i_1+1}$ by spoiler's set $A_{i_1+1}$ and the first-order quantifier by a set quantifier.
- Assume that spoiler plays a bound move, choosing $\mathcal{B}$ and bound $l \in \mathbb{N}$. Let

  $$\Phi_A = \left\{ \varphi \;\middle|\; \mathrm{rank}(\varphi) = k-1, \forall M \subseteq \mathcal{A} \left( l \leq |M| < \infty \Rightarrow \mathcal{A} \not\models \varphi(\bar{a}, \bar{A}, M) \right) \right\}.$$

  Note that $\mathcal{A} \models \mathsf{B} X \varphi(\bar{a}, \bar{A}, X)$ for all $\varphi \in \Phi_A$. Thus, $\mathcal{B} \models \mathsf{B} X \varphi(\bar{b}, \bar{B}, X)$ for all $\varphi \in \Phi_A$. Since $\Phi_A$ is finite up to equivalence we can fix a number $m \in \mathbb{N}$ that serves as a bound in $(\mathcal{B}, \bar{b}, \bar{B})$ for all $\varphi \in \Phi_A$. Thus, for the set

  $$\Phi_B = \left\{ \varphi \;\middle|\; \mathrm{rank}(\varphi) = k-1, \forall M \subseteq \mathcal{B} \left( m \leq |M| < \infty \Rightarrow \mathcal{B} \not\models \varphi(\bar{b}, \bar{B}, M) \right) \right\}$$

  we have $\Phi_A \subseteq \Phi_B$. Duplicator answers spoiler's challenge with this number $m$. Then spoiler has to choose a finite set $B_{i_2+1} \subseteq \mathcal{B}$ of size at least $m$. Let

  $$\Psi_B = \{\varphi \mid \mathrm{rank}(\varphi) = k-1, \mathcal{B} \models \varphi(\bar{b}, \bar{B}, B_{i_2+1})\}.$$

  Note that $\Phi_B \cap \Psi_B = \emptyset$. Since $\Psi_B$ is finite up to equivalence, there is a WMSO+B-formula $\psi \in \Psi_B$ of quantifier rank $k-1$ such that $\psi \equiv \bigwedge_{\varphi \in \Psi_B} \varphi$. In particular, $\psi \notin \Phi_B$. Hence, $\psi \notin \Phi_A$ (since $\Phi_A \subseteq \Phi_B$). By the definition of $\Phi_A$ this means that there is a finite subset $A_{i_2+1} \subseteq \mathcal{A}$ such that $|A_{i_2+1}| \geq l$ and $\mathcal{A} \models \psi(\bar{a}, \bar{A}, A_{i_2+1})$. Duplicator chooses this set $A_{i_2+1}$. The resulting position allows a winning strategy for duplicator by the induction hypothesis. ◻

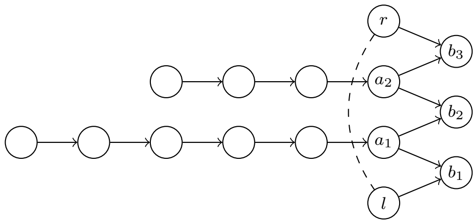


**Fig. 2** The standard $(5, 3)$-triple-u, where we only draw the Hasse diagram for $<^D$, and where dashed edges are $\perp$-edges.

## 6.2 The embeddable and the unembeddable triple-u structures

In this section we define a class of finite structures $\mathcal{G}_{n,m}$ for $n, m \in \mathbb{N}$. Using these structures, we define for every $k \geq 0$ the structures $\mathcal{E}_k$ and $\mathcal{U}_k$. We show that for every $k \geq 0$, $\mathcal{E}_k$ can be mapped homomorphically into a tree, whereas $\mathcal{U}_k$ cannot. In the next section, we will show that duplicator wins the $k$-round EF-game for both WMSO+B and MSO.

The *standard plain triple-u* is the structure $\mathcal{P} = (P, <^P, \perp^P)$, where

$$\begin{aligned} P &= \{l, r, a_1, a_2, b_1, b_2, b_3\}, \\ <^P &= \{(l, b_1), (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), (r, b_3)\}, \text{ and} \\ \perp^P &= \{(l, r), (r, l)\}. \end{aligned}$$

We call a structure $(V, <, \perp)$ a *plain triple-u* if it is isomorphic to the standard plain triple-u. For $n, m \in \mathbb{N}$, the *standard $(n, m)$-triple-u* is the structure

$$\begin{aligned} \mathcal{G}_{n,m} &= (D, <^D, \perp^D), \text{ where} \\ D &= \{l, r, a_1, a_2, b_1, b_2, b_3\} \cup (\{a_1\} \times \{1, 2, \ldots, n\}) \cup (\{a_2\} \times \{1, 2, \ldots, m\}), \end{aligned}$$

and $<^D, \perp^D$ are the minimal relations such that

- $\mathcal{G}_{n,m}$ restricted to $\{l, r, a_1, a_2, b_1, b_2, b_3\}$ is the standard plain triple-u,
- $(a_1, i) < (a_1, j)$ for all $1 \leq i < j \leq n$ and $(a_1, i) < a_1$ for all $1 \leq i \leq n$, and
- $(a_2, i) < (a_2, j)$ for all $1 \leq i < j \leq m$ and $(a_2, i) < a_2$ for all $1 \leq i \leq m$.

We call a graph $(V, <, \perp)$ an *$(n, m)$-triple-u* if it is isomorphic to the standard $(n, m)$-triple-u. Figure 2 depicts a $(5, 3)$-triple-u.

*Remark 22* For all $n, m \in \mathbb{N}$ and each $(n, m)$-triple-u $\mathcal{W}$ we fix an isomorphism $\psi_{\mathcal{W}}$ between $\mathcal{W}$ and the standard $(n, m)$-triple-u. Note that this isomorphism is unique if $n \neq m$. If $n = m$, then there is an automorphism of $\mathcal{G}_{n,n}$ exchanging the nodes $l$ and $r$. Thus, choosing an isomorphism means to choose the left node of the triple-u. For $x \in \{l, r, a_1, a_2, b_1, b_2, b_3\}$ we also write $\mathcal{W}.x$ for the node $\psi_{\mathcal{W}}^{-1}(x)$. Furthermore, we call the linear order of size $n$ (resp., $m$) that consists of all proper ancestors of $\mathcal{W}.a_1$ (resp., $\mathcal{W}.a_2$) the *left order* (resp., *right order*) of $\mathcal{W}$.

Let $k \in \mathbb{N}$ be a natural number. Fix a strictly increasing sequence $(n_{k,i})_{i \in \mathbb{N}}$ such that the linear order of length $n_{k,i}$ and the linear order of length $n_{k,j}$ are equivalent with respect to WMSO+B-formulas of quantifier rank up to $k$ for all $i, j \in \mathbb{N}$. Such a sequence exists because there are (up to equivalence) only finitely many WMSO+B-formulas of quantifier rank $k$. Since the linear orders of length $n_{k,i}$ are finite, they are equivalent with respect to both MSO-formulas and WMSO-formulas of quantifier rank up to $k$.

**Definition 23 (The embeddable triple-u)** Let $\mathcal{E}_k$ be the structure that consists of

1. the disjoint union of infinitely many $(n_{k,1}, n_{k,j})$-triple-u's and infinitely many $(n_{k,j}, n_{k,1})$-triple-u's for all $j \geq 2$,
2. one additional node $d$, and
3. for each triple-u $\mathcal{W}$ a $<$-edge from $\mathcal{W}.l$ to $d$.

In the following we call $d$ the *final node* of $\mathcal{E}_k$

**Lemma 24** *For all $k \in \mathbb{N}$, $\mathcal{E}_k$ admits a homomorphism to a tree.*

*Proof* Using the procedure on the central points from the ordinal tree setting described in the proof of Lemma 16, we first start adding the chains of each triple-u to the tree. In step $n_{k,1}$ we finally have placed all the chains of length $n_{k,1}$. Thus, for each triple-u $\mathcal{W}$ either $\mathcal{W}.a_1$ or $\mathcal{W}.a_2$ becomes central. Thus, in step $n_{k,1} + 1$ all the triple-u's split into two disconnected components and the incomparability edges, which were avoiding that $\mathcal{W}.l$ became central, now cease having such an effect. We can therefore map $\mathcal{W}.l$ (as well as $\mathcal{W}.r$) at stage $n_{k,1} + 2$ and the final node $d$ in step $n_{k,1} + 3$ to the tree. All the other nodes will be also mapped to the tree at some finite stage depending on the lengths of the chains $n_{k,i}$ for $i \geq 2$. Thus, it is easy to prove that the fixpoint procedure from the proof of Lemma 16 terminates at stage $\omega$. Whenever this happens, the given structure admits a homomorphism to a tree, see Remark 18. □

**Definition 25 (The unembeddable triple-u)** Let $\mathcal{U}_k$ be the structure that consists of

1. the disjoint union of infinitely many $(n_{k,j}, n_{k,j})$-triple-u's for all $j \geq 2$,
2. one additional node $d$, and
3. for each triple-u $\mathcal{W}$ a $<$-edge from $W.l$ to $d$.

In the following we call $d$ the *final node* of $\mathcal{U}_k$

**Lemma 26** *For all $k \in \mathbb{N}$, $\mathcal{U}_k$ does not admit a homomorphism to a tree.*

*Proof* Again, we consider the fixpoint procedure from the proof of Lemma 16. Assume that $\mathcal{U}_k$ admits a homomorphism to a tree. Then, the final node $d$ has to be placed at some stage $i$ into the tree, i.e., in the notation of the proof of Lemma 16, $d$ belongs to some set $C_i$ for $i < \omega$. But there is a $(n_{k,i}, n_{k,i})$-triple-u $\mathcal{W}$ and $\mathcal{W}.l < d$. Hence, $\mathcal{W}.l$ has to be placed into the tree in one of the first $i-1$ stages. But $\mathcal{W}.a_1$ and $\mathcal{W}.a_2$ are the target nodes of chains of length $n_{k,i} \geq i$. Hence, after $i$ stages they are still not mapped into the tree. Therefore, after $i$ stages, $\mathcal{W}.l$ and $\mathcal{W}.r$ are in the same connected component and they are linked by an $\perp$-edge. This contradicts the fact that $\mathcal{W}.l$ was placed into the tree in one of the first $i - 1$ stages. □

### 6.3 Duplicators strategies in the $k$-round game

We show that $\Theta$ does not have the EHD-property by showing that duplicator wins the $k$-round MSO-EF-game and WMSO+B-EF-game on the pair $(\mathcal{E}_k, \mathcal{U}_k)$ for each $k \in \mathbb{N}$. Hence, the two structures are not distinguishable by BMWB-formulas of quantifier rank $k$. For MSO this is rather simple. Since the linear orders of length $n_{k,i}$ and $n_{k,j}$ are indistinguishable up to quantifier rank $k$, it is straightforward to compile the strategies on these pairs of paths into a strategy on the whole structures for the $k$-round game. It is basically the same proof as the one showing that a strategy on a pair $(\biguplus_{i\in I} \mathcal{A}_i, \biguplus_{i\in I} \mathcal{B}_i)$ of disjoint unions can be compiled from strategies on the pairs $(\mathcal{A}_i, \mathcal{B}_i)$. In our situation there is an $i \in I$ such that $\mathcal{A}_i = \mathcal{B}_i$ consists of infinitely many plain triple-u's together with the final node, and the other pairs $(\mathcal{A}_j, \mathcal{B}_j)$ for $j \in I \setminus \{i\}$ consist of two linear orders that are indistinguishable by MSO-formulas of quantifier rank $k$. We leave the proof details as an exercise for the interested reader.

Compiling local strategies to a global strategy in the WMSO+B-EF-game is much more difficult because strategies are not closed under infinite disjoint unions. For instance, let $\mathcal{A}$ be the disjoint union of infinitely many copies of the linear order of size $n_{k,1}$ and $\mathcal{B}$ be the disjoint union of all linear orders of size $n_{k,j}$ for all $j \in \mathbb{N}$. Clearly, duplicator has a winning strategy in the $k$-round game starting on the pair that consists of the linear order of size $n_{k,1}$ and the linear order of size $n_{k,j}$. But in $\mathcal{A}$ every linear suborder has size bounded by $n_{k,1}$, while $\mathcal{B}$ has linear suborders of arbitrary finite size. This difference is of course expressible in WMSO+B. Even though strategies in WMSO+B-games are not closed under disjoint unions, we can obtain a composition result for disjoint unions on certain restricted structures as follows. Let $\mathcal{A} = \biguplus_{i\in\mathbb{N}} \mathcal{A}_i$ and $\mathcal{B} = \biguplus_{i\in\mathbb{N}} \mathcal{B}_i$ be disjoint unions of structures $\mathcal{A}_i$ and $\mathcal{B}_i$ satisfying the following conditions:

1. All $\mathcal{A}_i$ and $\mathcal{B}_i$ are finite structures.
2. For every $i \in \mathbb{N}$, duplicator has a winning strategy in the $k$-round MSO-EF-game on $\mathcal{A}_i$ and $\mathcal{B}_i$.
3. There is a constant $c \in \mathbb{N} \setminus \{0\}$ such that whenever spoiler starts the MSO-EF-game on $(\mathcal{A}_i, \mathcal{B}_i)$ with a set move choosing a set of size $n$ in $\mathcal{A}_i$ or $\mathcal{B}_i$, then duplicator's strategy answers with a set of size at least $\frac{n}{c}$.

In this case duplicator has a winning strategy in the $k$-round WMSO+B-EF-game on $\mathcal{A}$ and $\mathcal{B}$. To substantiate this claim, we sketch his strategy. For an element or set move, duplicator just uses the local strategies from the MSO-game to give an answer to any challenge. For a bound move, duplicator does the following. If spoiler's chooses the bound $l \in \mathbb{N}$, then duplicator chooses the number $m$, which is the total number of elements in all substructures $\mathcal{A}_i$ or $\mathcal{B}_i$ in which some elements have been chosen in one of the previous rounds plus $c \cdot l$. This forces spoiler to choose $c \cdot l$ elements in fresh substructures. Then duplicator uses his strategy in each local pair of structures to give an answer to spoiler's challenge. Since spoiler chose $c \cdot l$ elements in fresh substructures, duplicator answers with at least $\frac{c \cdot l}{c} = l$ many elements in fresh substructures. This is a valid move and it preserves the existence of local winning strategies between each pair $(\mathcal{A}_i, \mathcal{B}_i)$ for the rounds yet to play.

From now on, we consider a fixed number $k \in \mathbb{N}$ and the game on the structures $\mathcal{E}_k$ and $\mathcal{U}_k$. We use a variant of the closure under restricted disjoint unions, sketched above, to provide a winning strategy for duplicator. In order to reduce notational complexity we just write $\mathcal{E}$ for $\mathcal{E}_k$, $\mathcal{U}$ for $\mathcal{U}_k$ and $n_i$ for $n_{k,i}$ (for all $i \in \mathbb{N}$). With $\bar{E}$ (resp. $\bar{U}$) we denote the set of all maximal subgraphs that are $(n,m)$-triple-u's occurring in $\mathcal{E}$ (resp., $\mathcal{U}$) where $n$ and $m$ range over $\mathbb{N}$. Note that $\mathcal{E}$ is the disjoint union of all $W \in \bar{E}$ together with the final node, and similarly of $\mathcal{U}$. Unfortunately, we cannot apply the result on restricted disjoint unions directly because of the following problems.

- Due to the final nodes of $\mathcal{E}$ and $\mathcal{U}$, the structures are not disjoint unions of triple-u's. But since the additional structure in both structures is added in a uniform way this does not pose a problem for the proof.
- The greater cause for trouble is that there is no constant $c$ as in condition 3 that applies uniformly to all MSO-EF-games on an $(n_j, n_1)$-triple-u of $\mathcal{E}$ and an $(n_j, n_j)$-triple-u of $\mathcal{U}$ for all $j \in \mathbb{N}$. The problem is that if spoiler chooses in his first move all $n_j$ many elements of the right order of the $(n_j, n_j)$-triple-u, then the only possible answer of duplicator is to choose the set of the $n_1$ many elements of the right order of the $(n_j, n_1)$-triple-u. But since the numbers $n_j$ grow unboundedly, there is not constant $c$ such that the inequation $n_1 \geq \frac{n_j}{c}$ holds for all $j$. This problem does not exist for moves where spoiler chooses many elements in the left order of the $(n_j, n_j)$-triple-u. Duplicator's strategy allows to exactly choose the same subset of the left order of the $(n_j, n_1)$-triple-u. This allows to overcome the problem that duplicator should answer challenges where spoiler chooses a large set with an equally large set (up to some constant factor): Instead of assigning each triple-u in $\bar{E}$ a fixed corresponding triple-u in $\bar{U}$, we do this dynamically. If spoiler chooses a lot of elements from the left order of a fresh $(n_j, n_j)$-triple-u, then duplicator answers this challenge in a $(n_j, n_1)$-triple-u and we consider these two structures as forming one pair of the disjoint unions. On the other hand, if spoiler chooses a lot of elements from the right order of a fresh $(n_j, n_j)$-triple-u, then duplicators corresponding structure is chosen to be a fresh $(n_1, n_j)$-triple-u. In any case duplicator's local winning strategy may copy most of spoiler's choice (i.e., all elements chosen from the plain triple-u and from the order of length $n_j$ from which spoiler has chosen more elements), thus producing a set which is at least half as big as spoiler's challenge.

In our proof we encode this dynamic choice of corresponding structures as a partial map $\varphi : \bar{E} \to \bar{U}$. The following definition of a locally-$i$-winning position describes the requirements on a position obtained after playing some rounds that allow to further use local winning strategies in order to compile a winning strategy for the next $i$-rounds. It basically requires that the map $\varphi$ is such that for each triple-u $W \in \mathsf{dom}(\varphi)$ the restriction of the current position to $W$ and $\varphi(W)$ is a valid position in the $i$-round WMSO+B-EF-game on $(W, \varphi(W))$ which is winning for duplicator and that $\mathsf{dom}(\varphi)$ and $\mathsf{im}(\varphi)$ covers all elements that have been chosen so far (in an element move or as a member of some set).

**Definition 27** A position

$$p = (\mathcal{E}, e_1, \dots, e_{i_1}, E_1, \dots, E_{i_2}, \mathcal{U}, u_1, \dots, u_{i_1}, U_1, \dots, U_{i_2})$$

in the WMSO+B-EF-game on $(\mathcal{E}, \mathcal{U})$ is called locally-$i$-winning (for duplicator) if there is a partial bijection $\varphi : \bar{E} \to \bar{U}$ such that

- $\mathsf{dom}(\varphi)$ is finite,

- for all $W \in \bar{E}$, $W' \in \bar{U}$, and $1 \leq j \leq i_1$,
  1. if $e_j \in W$ then $W \in \mathsf{dom}(\varphi)$ and $u_j \in \varphi(W)$, and
  2. if $u_j \in W'$ then $W' \in \mathsf{im}(\varphi)$ and $e_j \in \varphi^{-1}(W')$,
- for all $W \in \bar{E}$, $W' \in \bar{U}$, and $1 \leq j \leq i_2$,
  1. if $E_j \cap W \neq \emptyset$ then $W \in \mathsf{dom}(\varphi)$ and
  2. if $U_j \cap W' \neq \emptyset$ then $W' \in \mathsf{im}(\varphi)$, and
- $\varphi$ is compatible with local strategies in the following sense:
  1. For all $W \in \mathsf{dom}(\varphi)$, $x \in \{l, r, a_1, a_2, b_1, b_2, b_3\}$, $1 \leq j \leq i_1$ and $1 \leq k \leq i_2$ we have
     - $e_j = W.x \Leftrightarrow u_j = \varphi(W).x$, and
     - $W.x \in E_k \Leftrightarrow \varphi(W).x \in U_k$.
  2. For all $W \in \mathsf{dom}(\varphi)$ and $1 \leq j \leq i_1$, $e_j$ belongs to the left (resp., right) order of $W$ if and only if $u_j$ belongs to the left (resp., right) order of $\varphi(W)$.
  3. For each $W \in \mathsf{dom}(\varphi)$, the restriction of the position $p$ to the left (resp., right) order of $W$ and the left (resp., right) order of $\varphi(W)$ is a winning position for duplicator in the $i$-round WMSO-EF-game.
  4. For all $1 \leq j \leq i_1$, $e_j$ is the final node of $\mathcal{E}$ if and only if $u_j$ is the final node of $\mathcal{U}$.
  5. For all $1 \leq j \leq i_2$, $E_j$ contains the final node of $\mathcal{E}$ if and only if $U_j$ contains the final node of $\mathcal{U}$.

*Remark 28* Note that the WMSO+B-EF-game on $(\mathcal{E},\mathcal{U})$ starts in a locally-$k$-winning position where the partial map $\varphi$ is the map with empty domain. Moreover, for all $i \in \mathbb{N}$, every locally-$i$-winning position is a winning position for duplicator in the 0-round WMSO+B-EF-game.

**Proposition 29** *Duplicator has a winning strategy in the $k$-round* WMSO+B-*EF-game on $(\mathcal{E}_k,\mathcal{U}_k)$.*

Due to the previous remark, the proposition follows directly form the following lemma.

**Lemma 30** *Let $1 \leq i \leq k$ be a natural number and $p$ a locally-$i$-winning position. Duplicator can respond any challenge of spoiler such that the next position is locally-$(i-1)$-winning.*

*Proof* Let $\varphi : \bar{E} \to \bar{U}$ be the partial bijection for the locally-$i$-winning position $p$. In the following, we say that an $(n,m)$-triple-u is *fresh* if it does not belong to $\mathsf{dom}(\varphi) \cup \mathsf{im}(\varphi)$. We consider the three possible types of moves for spoiler.

*Case 1.* Spoiler plays an element move. There are the following possibilities.

- If spoiler chooses the final node of one of the structures, duplicator answers with the final node of the other.
- If spoiler chooses some node from an $(n,m)$-triple-u $W \in \mathsf{dom}(\varphi)$, then the local strategy for $(W, \varphi(W))$ allows duplicator to answer this move with a node from $\varphi(W)$.
- Analogously, if spoiler chooses some node from an $(n,m)$- triple-u $W \in \mathsf{im}(\varphi)$, then the local strategy for $(\varphi^{-1}(W), W)$ allows duplicator to answer this move with a node from $\varphi^{-1}(W)$.

– If spoiler chooses a node from a fresh $(n, m)$-triple-u $W$ then duplicator can choose some fresh $(n', m')$-triple-u $W'$ from the other structure and can use the WMSO-equivalence up to quantifier rank $k$ of the left and right orders of $W$ and $W'$ to find a response to spoilers challenge such that adding $(W, W')$ (or $(W', W)$ depending on whether $W \in \bar{E}$) to $\varphi$ leads to a locally-$(i-1)$-winning position.

*Case 2.* Spoiler plays a set move. Then he chooses a finite set containing elements from some of the triple-u's from $\mathsf{dom}(\varphi)$ or $\mathsf{im}(\varphi)$ and from $l$ many fresh triple-u's. Choosing $l$ fresh triple-u's from the other structure, we can find a response on each of the triple-u's corresponding to the local strategy similar to the case of the element move. The union of all these local responses is a response for duplicator that leads to a locally-$(i-1)$-winning position.

*Case 3.* Spoiler plays a bound move. We distinguish on the structure he chooses.

– If he chooses structure $\mathcal{U}$ and the bound $l \in \mathbb{N}$, let $Z_n$ be the (finite) set of all $(n, n)$-triple-u's occurring in $\mathsf{im}(\varphi)$ and set

$$m_1 = \sum_{n \in \mathbb{N}} \sum_{W \in Z_n} (2n + 7).$$

Duplicator responds with the bound $m = m_1 + 2l$. Note that $2n+7$ is the size of an $(n, n)$-triple-u. Hence $m_1$ is the number of nodes in non-fresh triple-u's of $\mathcal{U}$. Next, spoiler chooses some finite subset $S$ of $\mathcal{U}$ with $|S| \geq m$. We construct a subset $S'$ in $\mathcal{E}$ such that the resulting position is locally-$(i-1)$-winning. Moreover, we guarantee that for any fresh triple-u $W \in \bar{U}$ such that $S \cap W \neq \emptyset$, duplicator's response $S' \cap W'$ in a corresponding fresh triple-u $W' \in \bar{E}$ contains at least $\frac{1}{2}|S \cap W|$ many elements. If $W_1, \ldots, W_z \in \bar{U}$ are all the fresh triple-u's that intersect $S$ non trivially, then we have $|\bigcup_{i=1}^{z}(W_i \cap S)| \geq m - m_1 - 1 = 2l - 1$ (the $-1$ comes from the fact that the final node of $\mathcal{U}$ may belong to $S$). Hence, duplicator's response $S'$ will contain at least $l$ many elements as desired. The concrete choice of $S'$ is done as follows.

(a) For all $W \in \mathsf{im}(\varphi)$, duplicator chooses a set $S'_W \subseteq \varphi^{-1}(W)$ such that $S'_W$ is the answer to spoiler's challenge $S \cap W$ according to a winning strategy in the $i$-round WMSO-EF-game on the restriction of $p$ to $\varphi^{-1}(W)$ and $W$. This winning strategy exists because position $p$ is locally $i$-winning.

(b) Now consider a fresh $(n, n)$-triple-u $W \in \bar{U}$ with $W \cap S \neq \emptyset$. Let $L$ (resp., $R$) be the nodes in the left (resp., right) order of $W$. If $|L \cap S| \geq |R \cap S|$, then take a fresh $(n, n_1)$-triple-u $W' \in \bar{E}$ (note that $n \geq n_1$) and extend the partial bijection $\varphi$ by $\varphi(W') = W$. Duplicator chooses the subset $S'_W = \psi(S \cap W \setminus R) \cup T$, where $\psi$ is the obvious isomorphism between the $(n, 0)$-sub-triple-u of $W$ (i.e., $W \setminus R$) and the $(n, 0)$-sub-triple-u of $W'$, and $T$ is an answer to spoilers move $S \cap R$ according to a winning strategy in the $i$-round WMSO-EF-game between the right order of $W'$ and the right order of $W$. Note that $|S'_W| \geq \frac{1}{2}|S \cap W|$.
If $|L \cap S| < |R \cap S|$, then let $W'$ be an $(n_1, n)$ triple-u and use the same strategy but reverse the roles of the left and the right order of the chosen triple-u's.

(c) If the final node of $\mathcal{U}$ is in $S$, let $S'_d$ be the singleton containing the final node of $\mathcal{E}$, otherwise let $S'_d = \emptyset$.

Finally, let $S'$ be the union of $S'_d$ and all sets $S'_W$ defined in (a) and (b) above. Since spoiler has chosen at least $2l-1$ many elements from fresh triple-u's, we directly conclude that $|S'| \geq l$. Moreover, since all the parts of $S'$ were defined using local strategies, we easily conclude that the position reached by choosing $S'$ is locally-$(i-1)$-winning.

– If spoiler chooses structure $\mathcal{E}$ and bound $l \in \mathbb{N}$, we use a similar strategy. Let $Y_n$ be the set of all $(n_1, n)$-triple-u's and all $(n, n_1)$-triple-u's occurring in $\mathsf{dom}(\varphi)$, and define

$$m_1 = \sum_{n \in \mathbb{N}} \sum_{W \in Y_n} n_1 + n + 7,$$

and $m_2 = l \cdot n_1$. Note that $m_1$ is the number of nodes from non-fresh triple-u's from $\mathcal{E}$. Duplicator responds with $m = m_1 + m_2 + l + 1$. Let $S \subseteq \mathcal{E}$ be spoiler's set with $|S| \geq m$. There are at least $m_2 + l$ elements in $S$ chosen from fresh triple-u's $W_1, W_2, \ldots, W_z \in \bar{E}$. Either $z > l$ or spoiler has chosen at least $l$ elements from $W_1 \cup W_2 \cup \cdots \cup W_z$ that do not belong to the orders of length $n_1$ (which in total contain only $z \cdot n_1 \leq l \cdot n_1 = m_2$ many elements). Duplicator chooses his response $S'$ in $\mathcal{U}$ as follows:

(a) For all $W \in \mathsf{dom}(\varphi)$, duplicator chooses a set $S'_W \subseteq \varphi(W)$ such that $S'_W$ is the answer to spoiler's challenge $S \cap W$ according to a winning strategy in the $i$-round WMSO-EF-game on the restriction of $p$ to $W$ and $\varphi(W)$. This winning strategy exists because position $p$ is locally $i$-winning.

(b) Now consider a fresh triple-u $W \in \bar{E}$ with $W \cap S \neq \emptyset$. If $W$ is an $(n_1, n)$-triple-u or an $(n, n_1)$-triple-u, let $W' \in \bar{U}$ be a fresh $(n, n)$-triple-u of $\mathcal{U}$, and extend the partial bijection $\varphi$ by $\varphi(W) = W'$. Let us consider the case that $W$ is an $(n, n_1)$-triple-u (for the other case one can argue analogously) and let $R$ be the right order (of size $n_1$) of $W$. Duplicator chooses the subset $S'_W = \psi(S \cap W \setminus R) \cup T$, where $\psi$ is the obvious isomorphism between the $(n, 0)$-sub-triple-u of $W$ (i.e., $W \setminus R$) and the $(n, 0)$-sub-triple-u of $W'$, and $T$ is an answer to spoiler's move $S \cap R$ according to a winning strategy in the $i$-round WMSO-EF-game between the right order of $W$ and the right order of $W'$. We can assume that $S'_W \neq \emptyset$, because we have $S \cap W \setminus R \neq \emptyset$ or $S \cap R \neq \emptyset$ and in the latter case $T$ can be chosen to be non-empty.

(c) If the final node of $\mathcal{E}$ is in $S$, let $S'_d$ be the singleton containing the final node of $\mathcal{U}$, otherwise let $S'_d = \emptyset$.

Finally, let duplicator's response $S'$ be the union of $S'_d$ and all sets $S'_W$ defined in (a) and (b) above. By the argument before (a), duplicator selects in (b) in total at least $l$ elements. Moreover, since all the parts of $S'$ where defined using local strategies, we easily conclude that the position reached by choosing $S'$ is locally-$(i-1)$-winning. ◻

Proposition 21 and 29 imply that for every $k \geq 1$, the structures $\mathcal{E}_k$ and $\mathcal{U}_k$ satisfy the same WMSO+B-sentences up to quantifier rank $k$. Moreover, these structures also satisfy the same MSO-sentences up to quantifier rank $k$, since as argued at the beginning of Section 6.3, duplicator can win the $k$-round MSO-EF-game on $(\mathcal{E}_k, \mathcal{U}_k)$. Since $\mathcal{E}_k$ admits a homomorphism to a tree (Lemma 24), whereas $\mathcal{U}_k$ does not (Lemma 26), it follows that there is no BMWB-sentence whose models are exactly those $\{<, \perp\}$-structures that allow a homomorphism to a tree. Hence, the EHD-method does not work for the class of all trees. Once again, note that this does not imply that satisfiability for ECTL* with constraints over the class of all trees (or, equivalently, the infinite tree $\mathcal{T}_\infty = (\mathbb{N}^*, <, \bot, =))$ is undecidable. In fact, as mentioned in the introduction, a recent work from Demri and Deters [9] established decidability of satisfiability for $\mathsf{CTL}^*$ with constraints over $\mathcal{T}_\infty$, and PSPACE-completeness of the corresponding LTL-fragment.

## 7 Open Problems

We conjecture that the above mentioned result of Demri and Deters [9] for $\mathsf{CTL}^*$ can be also extended to $\mathsf{ECTL}^*$ with constraints over $\mathcal{T}_\infty$. One might also ask whether there is some extension of the EHD-method that yields the result of Demri and Deters as a special instance of a more general meta theorem.

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## A A universal semi-linear order

In the following we define a semi-linear order which is universal for the class of all countable semi-linear orders. The fact that this order is universal is known to the experts in the field of semi-linear orders. Unfortunately, to our best knowledge there is no proof of this fact in the literature. Hence we provide a proof for this result.

**Definition 31** Let $\mathcal{U} = (U, <, \perp)$ be the countable semi-linear order with:

- $U = (\mathbb{N}\mathbb{Q})^*$ (the set of all finite sequences $n_1 q_1 \cdots n_k q_k$ with $k \geq 0$, $n_1, \dots, n_k \in \mathbb{N}$ and $q_1, \dots, q_k \in \mathbb{Q}$),
- $<$ is the strict order induced by $n_1 p_1 n_2 p_2 \cdots n_k p_k \leq m_1 q_1 m_2 q_2 \cdots m_l q_l$ iff
  - $k \leq l$, $n_i = m_i$ for all $1 \leq i \leq k$, $p_i = q_i$ for all $1 \leq i \leq k-1$ and $p_k \leq q_k$, and
  - $\perp = \perp_<$.

We call $\mathcal{U}$ the universal countable semi-linear order.

Note that Droste [13] has already studied this and similar orders.

For $u = n_1 p_1 n_2 p_2 \cdots n_k p_k \in U$ (with $k \geq 1$) and $q \in \mathbb{Q}$, we define

$$u + q = n_1 p_1 n_2 p_2 \cdots n_{k-1} p_{k-1} n_k (p_k + q). \tag{7}$$

We say that a countable semi-linear order $(A, <, \perp)$ is closed under finite infima if for each finite set $S \subseteq A$ the linear order $\{a \in A \mid a \leq s \text{ for all } s \in S\}$ has a maximal element, which is denoted by $\inf(S)$. Let $E = (a_i)_{i \in \mathbb{N}}$ be a repetition-free enumeration of $A$. We say $E$ is closed under infima if for each initial subset $A_i = \{a_1, a_2, \dots, a_i\}$ and each $S \subseteq A_i$ we have $\inf(S) \in A_i$.

**Lemma 32** *Let $\mathcal{A} = (A, <, \perp)$ be a countable semi-linear order. There is a countable semi-linear order $\mathcal{B}$ that is closed under finite infima and an injective homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

*Proof* For a nonempty subset $S \subseteq A$ we set $\downarrow S = \{a \in A \mid \forall s \in S\, (a \leq s)\}$. Let $\bar{A}$ be the set of finite nonempty subsets of $A$, which is obviously countable. We define an equivalence on $\bar{A}$ by setting $S \sim T$ iff $\downarrow S = \downarrow T$. For all $S \in \bar{A}$, $[S]$ denotes its equivalence class. Let $B$ be the set of all equivalence classes. We define an order $\sqsubset$ on $B$ by $[S] \sqsubset [T]$ if and only if $\downarrow S \subsetneq \downarrow T$.

We claim that $\mathcal{B} = (B, \sqsubset, \perp_\sqsubset)$ is a semi-linear order that is closed under finite infima and that the map $\varphi$ given by $\varphi(a) \mapsto [\{a\}]$ is an injective homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

- $\mathcal{B}$ is obviously a partial order. Moreover, note that $\downarrow S$ is a linear and downwards closed suborder of $\mathcal{A}$ for every nonempty finite set $S \subseteq A$. In order to show that $\mathcal{B}$ is semi-linear, assume that $[S_1] \sqsubset [S]$ and $[S_2] \sqsubset [S]$, i.e., $\downarrow S_1 \subsetneq \downarrow S$ and $\downarrow S_2 \subsetneq \downarrow S$. Thus, all elements from $\downarrow S_1$ and all elements from $\downarrow S_2$ are comparable. Since both sets are downwards closed, this directly implies that either $[S_1] = [S_2]$, $[S_1] \sqsubset [S_2]$ or $[S_2] \sqsubset [S_1]$.
- Let us show that $\mathcal{B}$ is closed under finite infima: Let $S, S_1, \dots, S_n$ be finite nonempty subsets of $A$ and assume that $[S] \sqsubseteq [S_i]$ for all $1 \leq i \leq n$. Thus, $\downarrow S \subseteq \downarrow S_i$. Hence, $\downarrow S \subseteq \bigcap_{i=1}^n \downarrow S_i = \downarrow \bigcup_{i=1}^n S_i$. Since $\downarrow \bigcup_{i=1}^n S_i \subseteq \downarrow S_i$ for all $1 \leq i \leq n$, $[\bigcup_{i=1}^n S_i] = \inf(\{[S_1], \dots, [S_n]\})$.

- For $a, b \in A$ with $a \neq b$ we have $b \notin \downarrow\{a\}$ or $a \notin \downarrow\{b\}$. Thus, $\varphi$ is an injective map from $A$ to $B$. Moreover, $a < b$ implies $\downarrow\{a\} \subsetneq \downarrow\{b\}$, i.e., $\varphi(a) \sqsubset \varphi(b)$. Similarly, $a \perp b$ implies $a \notin \downarrow\{b\}$ and $b \notin \downarrow\{a\}$, i.e., $\varphi(a) \perp_\sqsubset \varphi(b)$. ◻

**Lemma 33** *Let $\mathcal{A} = (A, <, \perp)$ be a countable semi-linear order that is closed under finite infima. There is a repetition-free enumeration of $\mathcal{A}$, which is closed under infima.*

*Proof* Fix an arbitrary repetition-free enumeration $(a_i)_{i\in\mathbb{N}}$ of $A$. Assume that we have constructed a sequence $b_1, b_2, \dots, b_i$ such that $B_j = \{b_1, b_2, \dots, b_j\}$ is closed under infima for every $j \leq i$. Let $k \in \mathbb{N}$ be minimal with $a_k \notin B_i$. Let $b'_1 < b'_2 < \dots b'_m < a_k$ be the list of all infima of the form $\inf(S \cup \{a_k\})$ for $S \subseteq B_i$ that are not contained in $B_i$. This list is indeed linearly ordered by $<$ since all elements in the list are bounded by $a_k$. Now set $b_{i+l} = b'_l$ for all $1 \leq l \leq m$ and set $b_{i+m+1} = a_k$. The resulting sequence $b_1, \dots, b_{i+m+1}$ contains $a_k$ and $B_j = \{b_1, b_2, \dots, b_j\}$ is closed under infima for every $j \leq i + m + 1$. This can be easily shown using the fact that $\inf(X \cup \{\inf(Y)\}) = \inf(X \cup Y)$ for all sets $X$ and $Y$.

Repeating this construction leads to an enumeration $(b_i)_{i\in\mathbb{N}}$ of $A$ with the desired property. ◻

**Lemma 34** *Let $\mathcal{A} = (A, \sqsubset, \perp_\sqsubset)$ be a countable semi-linear order. There exists an injective homomorphism from $\mathcal{A}$ into $\mathcal{U}$.*

*Proof* Due to Lemma 32 and 33, we may assume that $\mathcal{A}$ is closed under finite infima and that $(a_i)_{i\in\mathbb{N}}$ is a repetition-free enumeration of $A$ which is closed under finite infima. Set $A_i = \{a_1, \dots, a_i\}$ for $i \geq 1$. Inductively, we construct injective homomorphisms $\varphi_i : A_i \to U$ ($i \geq 1$) such that

1. $\varphi_{i+1}$ extends $\varphi_i$, and
2. for all $u = n_1p_1n_2p_2 \dots n_kp_k \in \mathsf{im}(\varphi_i)$ and all $1 \leq j \leq k$ we have $p_j \in \frac{1}{2^i}\mathbb{Z}$.

Define $\varphi_1 : A_1 \to U$ by $\varphi_1(a_1) = 00 \in \mathbb{N}\mathbb{Q}$. Assume that $\varphi_i$ has already been constructed. We distinguish two cases.

1. If there is some $a \in A_i$ with $a_{i+1} \sqsubset a$ let $u = \inf\{a \in A_i \mid a_{i+1} \sqsubset a\}$. Note that $a_{i+1} \sqsubseteq u$. Since the enumeration is closed under infima, we have $u \in A_i$ (and thus $a_{i+1} \sqsubset u$) and we can define $\varphi_{i+1}(a_{i+1}) = \varphi_i(u) + (\frac{-1}{2^{i+1}})$, where we add according to (7). Note that $\varphi_{i+1}(a_{i+1}) < \varphi_i(u) = \varphi_{i+1}(u)$. In order to prove that this defines a homomorphism, we distinguish the following cases:
   (a) If $a_{i+1} \sqsubset a$ for some $a \in A_i$ then $u \sqsubseteq a$. Hence, $\varphi_{i+1}(a_{i+1}) < \varphi_{i+1}(u) = \varphi_i(u) \leq \varphi_i(a) = \varphi_{i+1}(a)$ as desired.
   (b) If $a \sqsubset a_{i+1}$ for some $a \in A_i$, then $a \sqsubset u$. Hence, $\varphi_{i+1}(a) = \varphi_i(a) < \varphi_i(u)$. Since $\varphi_i$ uses only rationals from $\frac{1}{2^i}\mathbb{Z}$, we conclude that $\varphi_{i+1}(a) \leq \varphi_i(u) + \frac{-1}{2^i} < \varphi_{i+1}(a_{i+1})$ as desired.
   (c) If $a_{i+1} \perp_\sqsubset a$ for some $a \in A_i$, then $a \perp_\sqsubset u$. By induction, $\varphi_{i+1}(a) = \varphi_i(a) \perp_< \varphi_i(u) = \varphi_{i+1}(u)$. Thus, the assumption $\varphi_{i+1}(a) \leq \varphi_{i+1}(a_{i+1})$ leads by transitivity of $\leq$ to the contradiction $\varphi_{i+1}(a) \leq \varphi_{i+1}(u)$. Similarly, the assumption $\varphi_{i+1}(a) > \varphi_{i+1}(a_{i+1})$ yields $\varphi_{i+1}(a) \geq \varphi_{i+1}(a_{i+1}) + \frac{1}{2^{i+1}} = \varphi_{i+1}(u)$. We can conclude that $\varphi_{i+1}(a) \perp_< \varphi_{i+1}(a_{i+1})$ as desired.
2. Otherwise, for all $j \leq i$ we know that $\inf\{a_j, a_{i+1}\}$ is strictly below $a_{i+1}$ and hence belongs to $A_i$ (since the enumeration is closed under infima). In particular, the set $\{a \in A_i \mid a < a_{i+1}\}$ is not empty. By semi-linearity, $u = \max\{a \in A_i \mid a < a_{i+1}\}$ is well-defined. Since $\mathsf{im}(\varphi_i)$ is finite, there is some $n \in \mathbb{N}$ such that $\varphi_i(u)n0$ is incomparable to all elements from the set $\varphi_i(A_i \setminus \{a \in A_i \mid a \leq u\})$. Extending $\varphi_i$ by setting $\varphi_{i+1}(a_{i+1}) = \varphi_i(u)n0$ is easily shown to be a homomorphism.

Finally, the limit of $(\varphi_i)_{i\in\mathbb{N}}$ clearly defines an injective homomorphism from $\mathcal{A}$ into $\mathcal{U}$. ◻