Exponent equations in HNN-extensions

Michael Figelius
Markus Lohrey
figelius@eti.uni-siegen.de
lohrey@eti.uni-siegen.de
University of Siegen
Siegen, Germany

ABSTRACT

We consider exponent equations in finitely generated groups. These are equations, where the variables appear as exponents of group elements and take values from the natural numbers. Solvability of such (systems of) equations has been intensively studied for various classes of groups in recent years. In many cases, it turns out that the set of all solutions on an exponent equation is a semilinear set that can be constructed effectively. Such groups are called knapsack semilinear. The class of knapsack semilinear groups is quite rich and it is closed under many group theoretic constructions, e.g., finite extensions, graph products, wreath products, amalgamated free products with finite amalgamated subgroups, and HNN-extensions with finite associated subgroups. On the other hand, arbitrary HNN-extensions do not preserve knapsack semilinearity. In this paper, we consider the knapsack semilinearity of HNN-extensions, where the stable letter t acts trivially by conjugation on the associated subgroup A of the base group G. We show that under some additional technical conditions, knapsack semilinearity transfers from the base group G to the HNN-extension. These additional technical conditions are satisfied in many cases, e.g., when A is a centralizer in G or A is a quasiconvex subgroup of the hyperbolic group G.

CCS CONCEPTS

• **Theory of computation** → *Formal languages and automata theory*.

KEYWORDS

algorithmic group theory, equations in groups, HNN-extensions, hyperbolic groups

ACM Reference Format:
Michael Figelius and Markus Lohrey. 2022. Exponent equations in HNN-extensions. In *Proceedings of (ISSAC 2022).* ACM, New York, NY, USA, 9 pages. https://doi.org/XXXXXXX.XXXXXXX

This work has been supported by the DFG research project LO 748/13-1.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.
ISSAC 2022, July 04–07, 2018, Lille, France
© 2022 Association for Computing Machinery.
ACM ISBN 978-1-4503-XXXX-X/18/06…$15.00
https://doi.org/XXXXXXX.XXXXXXX

1 INTRODUCTION

For an infinite finitely generated group G and so-called exponent variables $x_1, \dots, x_k$ we consider equations of the form

$$h_0 g_1^{x_1} h_1 g_1^{x_2} \cdots h_{k-1} g_k^{x_k} h_k = 1 \tag{1}$$

where the g_i and h_i are elements of G (given as words over a generating set) and the every x_i ranges over $\mathbb{N}$. Equations of this form are known as *knapsack equations* and have received a lot of attention in recent years, see e.g. [1–5, 8, 10–12, 22–25, 27, 28]. Note that in a knapsack equation, the exponent variables are assumed to be pairwise different. If this is not assumed, i.e., if $x_i = x_j$ for $i \neq j$ is allowed, then one speaks of an *exponent equation*. Several variants and problems have been studied in this context. The most general decision problem is to decide whether a given system of exponent equations has a solution where natural numbers are assigned to the variables x_i. This problem is known to be decidable in hyperbolic groups [23], virtually special groups (finite extensions of subgroups of right-angled Artin groups)[1] [24], co-context-free groups (groups where the complement of the word problem is context-free) [22], and free solvable groups [8]. A simpler problem is the so-called knapsack problem, where it is asked whether a single knapsack equation has a solution. There are groups, with a decidable knapsack problem, but an undecidable solvability problem for systems of exponent equations. Examples are the discrete Heisenberg group [22] and the Baumslag-Solitar group $\mathsf{BS}(1,2)$ [2, Theorem E.1]. Let us also remark that the variants of these problems, where the variables x_i range over $\mathbb{Z}$ are not harder (one can replace a power $g_i^{x_i}$ with $x_i \in \mathbb{Z}$ by $g_i^{x_i} (g_i^{-1})^{y_i}$ with $x_i, y_i \in \mathbb{N}$).

Another problem is to describe the set of all solutions of a knapsack equation. It turned out that for many groups this set is effectively semilinear for every knapsack equation;[2] such groups are called *knapsack semilinear*. For instance, the above mentioned groups (hyperbolic groups, virtually special groups, co-context-free groups and free solvable groups) are knapsack semilinear and the class of knapsack semilinear groups is closed under the following operations: finite extensions [10], graph products [10], wreath products [12], amalgamated free products with finite amalgamated subgroups [10], and HNN-extensions with finite associated subgroups [10]. For a knapsack semilinear group one can decide whether a given system of exponent equations has a solution. Moreover, for

[1]Many groups are known to be virtually special, e.g., Coxeter groups, fully residually free groups, one-relator groups with torsion, and fundamental groups of hyperbolic 3-manifolds.

[2]A subset of $\mathbb{N}^n$ is semilinear if it is a finite union of sets of shifted subsemigroups of $(\mathbb{N}^n, +)$; see Section 2.1 for more details. Effectively semilinear means that the finitely many vectors from a semilinear description of the solution set of (1) can be computed from words representing the group elements g_i and h_i in (1).

a knapsack semilinear group also the set of all solutions of an exponent equations is effectively semilinear; this follows easily from well-known closure properties of semilinear sets.

In this paper we want to further elaborate HNN-extensions. HNN-extension is a fundamental operation in all areas of geometric and combinatorial group theory. A theorem of Seifert and van Kampen links HNN-extensions to algebraic topology, see e.g. [34, p. 407]. Moreover, HNN-extensions are used in all modern proofs for the undecidability of the word problem in finitely presented groups; see e.g. [36, Section 9.3]. For a base group G with two isomorphic subgroups A and B and an isomorphism $\varphi\colon A \to B$, the corresponding HNN-extension is the group

$$H = \langle G, t \mid t^{-1}at = \varphi(a)\ (a \in A)\rangle. \tag{2}$$

Intuitively, it is obtained by adjoining to G a new generator t (the *stable letter*) in such a way that conjugation of A by t realizes φ. The subgroups A and B are also called the *associated subgroups*. Recall from the above discussion that if G is knapsack semilinear and A and B are finite then also H is knapsack semilinear [10]. For arbitrary HNN-extensions, this is not true. For instance, the Baumslag-Solitar group $\mathsf{BS}(1,2) = \langle a, t \mid t^{-1}at = a^2\rangle$ is not knapsack semilinear [25] but it is an HNN-extension of the knapsack semilinear group $\langle a\rangle \cong \mathbb{Z}$. This example shows that we have to drastically restrict HNN-extensions in order to get a transfer result for knapsack semilinearity beyond the case of finite associated subgroups. In this paper we study HNN-extensions of the form

$$H = \langle G, t \mid t^{-1}at = a\ (a \in A)\rangle, \tag{3}$$

where $A \leq H$ is a subgroup. In other words, we take in (2) for $\varphi : A \to B$ the identity on A. Intuitively: we add to the group G a free generator t together with commutation identities $at = ta$ for all $a \in A$. This operation interpolates between the free product $G * \langle t\rangle \cong G * \mathbb{Z}$ and the direct product $G \times \langle t\rangle \cong G \times \mathbb{Z}$.

Even HNN-extensions of the form (3) with f.g. A are too general for our purpose: if the subgroup membership problem for A is undecidable then H has an undecidable word problem. Hence, we also need some restriction on the subgroup $A \leq G$.

Definition 1.1. We say that G is knapsack semilinear relative to the subset $S \subseteq G$ if for all $h_0, g_1, h_1, g_2, \ldots, h_{k-1}, g_k, h_k \in G$ the set of all tuples $(n_1, \ldots, n_k) \in \mathbb{N}^n$ with $h_0 g_1^{n_1} h_1 g_1^{n_2} \cdots h_{k-1} g_k^{n_k} h_k \in S$ is effectively semilinear (we are mainly interested in the case where S is a subgroup of G). For sets $S_1, \ldots, S_k \subseteq G$ we say that G is knapsack semilinear relative to $\{S_1, \ldots, S_k\}$ if for every $1 \leq i \leq k$, G is knapsack semilinear relative to S_i.

Note that G is knapsack semilinear iff it is knapsack semilinear relative to 1. Our first main result is:

Theorem 1.2. *If G is knapsack semilinear relative to $\{1, A\}$ then $H = \langle G, t \mid t^{-1}at = a\ (a \in A)\rangle$ is knapsack semilinear.*

In some situations we can even avoid the explicit assumption that G is knapsack semilinear relative to the subgroup A. HNN-extensions of the form (3), where A is the centralizer of a single element $g \in G$ are known as *free rank one extensions of centralizers* and were first studied in [29] in the context of so-called exponential groups. It is easy to observe that if G is knapsack semilinear and $A \leq G$ is the centralizer of a finite set of elements, then G is also knapsack semilinear relative to A. In particular the operation of free rank one extension of centralizers preserves knapsack semilinearity. A corollary of this result is that every fully residually free group is knapsack semilinear. The class of fully residually free groups is exactly the class of all groups that can be constructed from $\mathbb{Z}$ by the following operations: taking finitely generated subgroups, free products and free rank one extensions of centralizers. Knapsack semilinearity of fully residually free groups also follows from the fact that every fully residually free group is virtually special [37].

In the second part of the paper, we study HNN-extensions of the form (3), where G is a hyperbolic group. A group is hyperbolic if all geodesic triangles in the Cayley-graph are δ-slim for a constant δ. The class of hyperbolic groups has several alternative characterizations (e.g., it is the class of finitely generated groups with a linear Dehn function), which gives hyperbolic groups a prominent role in geometric group theory. Moreover, in a certain probabilistic sense, almost all finitely presented groups are hyperbolic [14, 31]. Also from a computational viewpoint, hyperbolic groups have nice properties: it is known that the word problem and the conjugacy problem can be solved in linear time [7, 18]. In [23] it was shown that hyperbolic groups are knapsack semilinear. Here we extend this result by showing the following:

Theorem 1.3. *Let G be hyperbolic and let H be a quasiconvex subgroup of G. Then G is knapsack semilinear relative to H.*

Quasiconvex subgroups in hyperbolic groups are known to have nice properties. Many algorithmic problems are decidable for quasiconvex subgroups, including the membership problem [21], whereas Rips constructed finitely generated subgroups of hyperbolic groups with an undecidable membership problem [33].

A more detailed version of this paper can be found in [9].

2 PRELIMINARIES

In the following three subsections we introduce some definitions concerning semilinear sets, finite automata, and groups.

2.1 Semilinear sets

Fix a dimension $d \geq 1$. A *linear subset* of $\mathbb{N}^d$ is a set of the form $L(b_0, \ldots, b_k) = \{b_0 + a_1 b_1 + \cdots a_k b_k \mid a_1, \ldots, a_k \in \mathbb{N}\}$ with $b_0, \ldots, b_k \in \mathbb{N}^d$. A subset $S \subseteq \mathbb{N}^d$ is *semilinear*, if it is a finite union of linear sets. The class of semilinear sets is known to be effectively closed under boolean operations, projections, and pointwise addition. For a semilinear set $S = \bigcup_{1 \leq i \leq n} L(b_{i,0}, \ldots, b_{i,k_i})$, we call the tuple $(b_{i,j})_{1 \leq i \leq n, 1 \leq j \leq k_i}$ a *semilinear representation* of S.

The semilinear sets are exactly those sets that are definable in first-order logic over the structure $(\mathbb{N}, +)$ (the so-called *Presburger definable sets*). All the known closure properties of semilinear sets follow from this characterization. A good survey on semilinear results and Presburger arithmetic with references for the above mentioned results is [15].

2.2 Regular languages and rational relations

More details on finite automata can be found in the standard text book [20]. Let Σ be a finite alphabet of symbols. As usual, Σ^* denotes the set of all finite words over the alphabet Σ. For a word $w = a_1 a_2 \cdots a_n$ with $a_1, \ldots, a_n \in \Sigma$ we denote with $|w| = n$ the length of w. We denote the empty word (the unique word of length 0) with ε; in group theoretic contexts we also write 1 for the empty word.

A *finite automaton* over the alphabet Σ is a tuple $\mathcal{A} = (Q, I, \delta, F)$, where Q is a finite set of states, $I \subseteq Q$ is the set of initial states, $\delta \subseteq Q \times \Sigma \times Q$ is the set of transitions, and $F \subseteq Q$ is the set of final states. A word $w = a_1 a_2 \cdots a_n$ is accepted by $\mathcal{A}$ if there are transitions $(q_{i-1}, a_i, q_i) \in \delta$ for $1 \leq i \leq n$ such that $q_0 \in I$ and $q_n \in F$. With $L(\mathcal{A})$ (the language accepted by $\mathcal{A}$) we denote the set of all words accepted by $\mathcal{A}$. A language L is called *regular* if it is accepted by a finite automaton.

We fix an arbitrary enumeration $a_1, \ldots, a_k$ of the alphabet Σ. For $w \in \Sigma^*$ and $1 \leq i \leq k$ let $|w|_{a_i}$ be the number of occurrences of a_i in w. The *Parikh image* of w is $P(w) = (|w|_{a_1}, \ldots, |w|_{a_k}) \in \mathbb{N}^k$. The Parikh image of a language $L \subseteq \Sigma^*$ is $P(L) = \{P(w) \mid w \in L\}$. The following important result was shown by Parikh [32].

Theorem 2.1. *The semilinear sets are exactly the Parikh images of the regular languages. From a given finite automaton $\mathcal{A}$ one can compute a semilinear representation of $P(L(\mathcal{A}))$.*

We will also use the following simple lemma:

Lemma 2.2 (c.f. [10, Lemma 5.8]). *Let $p, q, r, s, u, v \in \Sigma^*$. Then the set $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid pq^x r = su^y v\}$ is semilinear and a semilinear representation can be computed from p, q, r, s, u, v.*

A *finite state transducer* $\mathcal{T}$ over the alphabet Σ is a tuple $\mathcal{T} = (Q, I, \delta, F)$ where Q, I and F have the same meaning as in a finite automaton and $\delta \subseteq Q \times ((\Sigma \times \{\varepsilon\}) \cup (\{\varepsilon\} \times \Sigma)) \times Q$. A pair $(u, v) \in \Sigma^* \times \Sigma^*$ is accepted by $\mathcal{T}$ if there are transitions $(q_{i-1}, a_i, b_i, q_i) \in \delta$ for $1 \leq i \leq |u| + |v|$ $(a_i, b_i \in \Sigma \cup \{\varepsilon\})$ such that $u = a_1 \cdots a_{|u|+|v|}$, $v = b_1 \cdots b_{|u|+|v|}$, $q_0 \in I$ and $q_{|u|+|v|} \in F$. With $R(\mathcal{T})$ we denote the set of all pairs accepted by $\mathcal{T}$. A relation $R \subseteq \Sigma^* \times \Sigma^*$ is a *rational relation* if it is accepted by a finite state transducer.

2.3 Groups

For more details on group theory we refer to [26]. Infinite groups are usually given by presentations. Take a non-empty set Ω and let $\Omega^{-1} = \{a^{-1} \mid a \in \Omega\}$ be a set of formal inverses such that $\Omega \cap \Omega^{-1} = \emptyset$. Let $\Sigma = \Omega \cup \Omega^{-1}$. On the set Σ^* there is a natural involution $(\cdot)^{-1}$ defined by $(a^{-1})^{-1} = a$ for $a \in \Omega$ and $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$ for $a_1, \ldots, a_n \in \Sigma$. A word $w \in \Sigma^*$ is called *reduced* if it does not contain an occurrence of a word aa^{-1} or $a^{-1}a$ $(a \in \Sigma)$. Applying the cancellation rules $aa^{-1} \to \varepsilon$ or $a^{-1}a \to \varepsilon$ as long as possible, every word $w \in \Sigma^*$ can be mapped to a unique reduced word $\mathrm{red}(w)$. The *free group* $F(\Sigma)$ consists of all reduced words together with the group multiplication $u \cdot v = \mathrm{red}(uv)$ for reduced words u, v. The mapping red can be also viewed as a monoid morphism from Σ^* to $F(\Sigma)$. For a subset $R \subseteq \Sigma^*$ one defines the group $\langle \Sigma \mid R \rangle$ as the quotient group $F(\Sigma)/N_R$, where N_R is the intersection of all normal subgroups of $F(\Sigma)$ that contain $\mathrm{red}(R)$ (the normal closure of R). Clearly, every group is isomorphic to a group of the form $\langle \Sigma \mid R \rangle$.

Let $G = \langle \Sigma \mid R \rangle$ in the following. If Σ is finite then G is called *finitely generated* (f.g. for short) and Σ is called a *finite symmetric generating set* for G. If both Σ and R are finite, then G is called *finitely presented*. The surjective monoid morphism $\mathrm{red}\colon \Sigma^* \to F(\Sigma)$ extends to a surjective monoid morphism $h\colon \Sigma^* \to G$, called the *evaluation morphism*. For two words $u, v \in \Sigma^*$ we write $u =_G v$ if $h(u) = h(v)$. For a subset $S \subseteq G$ we write $u \in_G S$ if $h(u) \in S$.

3 KNAPSACK AND EXPONENT EQUATIONS

Let G be a f.g. group with the finite symmetric generating set Σ. Recall the notion of knapsack semilinearity relative to a subset $S \subseteq G$ from Definition 1.1. If G is knapsack semilinear relative to 1 then G is called knapsack semilinear. All group elements in a knapsack (or exponent) equation will be given by words over the alphabet Σ.

We introduce a few additional notations for knapsack and exponent equations. Fix pairwise different exponent variables $x_1, \ldots, x_\ell$. An *exponent expression* over Σ is a formal expression of the form

$$e = v_0 u_1^{y_1} v_1 u_2^{y_2} \cdots v_{k-1} u_k^{y_k} v_k \tag{4}$$

with $k \geq 1$, words $u_i, v_i \in \Sigma^*$ and $y_1, \ldots, y_k \in \{x_1, \ldots, x_\ell\}$. We also write $e(x_1, \ldots, x_\ell)$ in order to make the exponent variables explicit. For natural numbers $n_1, \ldots, n_\ell \in \mathbb{N}$ let $e(n_1, \ldots, n_\ell) \in \Sigma^*$ be the word obtained by replacing in e every exponent variable x_i by n_i. A tuple $(n_1, \ldots, n_\ell)$ is called a *G-solution* (or simply a solution if G is clear from the context) of the knapsack equation $e = 1$ if $e(n_1, \ldots, n_\ell) =_G 1$ holds. We can assume that $u_i \neq \varepsilon$ for all $1 \leq i \leq k$. If all exponent variable in (4) are pairwise different then e is called a knapsack expression. If G is knapsack semilinear, then also for every exponent expression e the set of all G-solutions of $e = 1$ is effectively semilinear. This is a consequence of the effective closure properties of semilinear sets; see e.g. [10].

As mentioned in the introduction, the class of knapsack semilinear groups is very rich. Groups that are not knapsack semilinear are the 3-dimensional Heisenberg group $H_3(\mathbb{Z})$ [22] and the Baumslag-Solitar group $\mathrm{BS}(1,2)$ [2, 25]. These groups are not knapsack semilinear in a strong sense: there are knapsack expressions e such that the set of all $H_3(\mathbb{Z})$-solutions (resp., $\mathrm{BS}(1,2)$-solutions) of $e = 1$ is not semilinear.

4 HNN-EXTENSIONS

In this section we introduce HNN-extensions. Suppose $G = \langle \Sigma \mid R \rangle$ is a f.g. group with the finite symmetric generating set $\Sigma = \Omega \cup \Omega^{-1}$ and $R \subseteq \Sigma^*$. Fix two isomorphic subgroups A and B of G together with an isomorphism $\varphi\colon A \to B$. Let $t \notin \Sigma$ be a new letter. Then the corresponding *HNN-extension* is the group

$$H = \langle \Sigma \cup \{t, t^{-1}\} \mid R \cup \{t^{-1}a^{-1}t\varphi(a) \mid a \in A\} \rangle$$

(formally, we identify here every element $g \in A \cup B$ with a word over Σ that evaluates to g). This group is usually denoted by

$$H = \langle G, t \mid t^{-1}at = \varphi(a) \ (a \in A) \rangle. \tag{5}$$

Intuitively, H is obtained from G by adding a new element t such that conjugating elements of A with t applies the isomorphism φ. Here, t is called the *stable letter* and the groups A and B are the *associated subgroups*. A basic fact about HNN-extensions is that the group G embeds naturally into H [17].

In this paper, we consider HNN-extensions $H = \langle G, t \mid t^{-1}at = \varphi(a) \ (a \in A) \rangle$, where $A \leq G$ is a subgroup of $G = \langle \Sigma \mid R \rangle$ and $\varphi : A \to A$ is the identity mapping. Thus, H can be written as

$$H = \langle G, t \mid t^{-1}at = a \ (a \in A) \rangle. \tag{6}$$

Let us fix this HNN-extension for the further consideration. Let us denote with $h : (\Sigma \cup \{t, t^{-1}\})^* \to H$ the evaluation morphism.

A word $u \in (\Sigma \cup \{t, t^{-1}\})^*$ is called *Britton-reduced* if it does not contain a factor of the form $t^{-\alpha} w t^{\alpha}$ with $\alpha \in \{-1, 1\}$, $w \in \Sigma^*$ and $w \in_G A$. A factor of the form $t^{-\alpha} w t^{\alpha}$ with $\alpha \in \{-1, 1\}$, $w \in \Sigma^*$ and $w \in_G A$ is also called a *pin*, which we can replace by w. Since this decreases the number of t's in the word, we can reduce every word to an equivalent Britton-reduced word. We denote the set of all Britton-reduced words in the HNN-extension (6) by $\mathsf{BR}(H)$. For $u \in (\Sigma \cup \{t, t^{-1}\})^*$ we define $\pi_t(u)$ as the projection of the word u onto the alphabet $\{t, t^{-1}\}$ and $\pi_\Sigma(u)$ as the projection of the word u into the monoid Σ^*. Britton's lemma states that if $u =_H 1$ ($u \in (\Sigma \cup \{t, t^{-1}\})^*$) then u contains a pin or $u \in \Sigma^*$ and $u =_G 1$. Note that a consequence of this is that if $u \in_H G$ (which means that $uv =_H 1$ for a word $v \in \Sigma^*$), then u contains a pin or $u \in \Sigma^*$. In particular a Britton-reduced word that contains $t^{\pm 1}$ cannot represent an element of the base group G.

Note that $H/N(t) \cong G$, where $N(t)$ is the normal subgroup generated by t. By $\pi_G : H \to G$ we denote the canonical projection. We have $\pi_G(g_0 t^{\delta_1} g_1 \cdots t^{\delta_k} g_k) = g_0 g_1 \cdots g_k$ for $g_0, \ldots, g_k \in G$. Hence, on the level of words, π_G is computed by the projection $\pi_\Sigma : (\Sigma \cup \{t, t^{-1}\})^* \to \Sigma^*$.

Lemma 4.1. *Let $w \in (\Sigma \cup \{t, t^{-1}\})^*$. Then $w =_H 1$ if and only if $w \in_H G$ and $\pi_\Sigma(w) =_G 1$.*

Proof. If $w =_H 1$ then $w \in_H G$. Moreover, by Britton's lemma, w can be reduced to a word from Σ^* using Britton reduction. But this word must be $\pi_\Sigma(w)$, which implies $\pi_\Sigma(w) =_G 1$. On the other hand, if $w \in_H G$ and $\pi_\Sigma(w) =_G 1$, then, again, w can be reduced to $\pi_\Sigma(w) =_G 1$ using Britton reduction, which yields $w =_H 1$. ◻

We will also need the following lemma; see [16, Lemma 2.3].

Lemma 4.2. *Let $u = u_0 t^{\delta_1} u_1 \cdots t^{\delta_k} u_k$ and $v = v_0 t^{\varepsilon_1} v_1 \cdots t^{\varepsilon_\ell} v_\ell$ be Britton-reduced words with $u_i, v_j \in \Sigma^*$. Let $0 \le r \le \max\{k, \ell\}$ be the largest number such that*

- *$\delta_{k-i} = -\varepsilon_{i+1}$ for all $0 \le i \le r-1$ and*
- *$u_{k-i+1} \cdots u_k v_0 \cdots v_{i-1} \in_G A$ for all $0 \le i \le r$ (for $i = 0$ this condition is trivially satisfied).*

Then $w = u_0 t^{\delta_1} u_1 \cdots t^{\delta_{k-r}} u_{k-r} \cdots u_k v_0 \cdots v_r t^{\varepsilon_{r+1}} v_{r+1} \cdots t^{\varepsilon_\ell} v_\ell$ is a Britton-reduced word with $w =_H uv$.

5 KNAPSACK SEMILINEAR HNN-EXTENSIONS

In this section we show that the HNN-extension (6) is knapsack semilinear if G is knapsack semilinear relative to $\{1, A\}$. We first introduce some concepts that can be found in a similar form in [10].

We call a word $w \in \mathsf{BR}(H)$ *well-behaved*, if w^n is Britton-reduced for every $n \ge 0$. Note that w is well-behaved if and only if w and w^2 are Britton-reduced. Every word $w \in \Sigma^*$ is well-behaved.

Lemma 5.1 (c.f. [10, Lemma 6.3]). *From a given word $u \in \mathsf{BR}(H)$ we can compute words $s, p, v \in \mathsf{BR}(H)$ such that $u^n =_H s v^n p$ for every $m \ge 0$ and v is well-behaved.*

In the following we assume that G is knapsack semilinear relative to $\{1, A\}$. Let $e(x_1, \ldots, x_n) = v_0 u_1^{x_1} \cdots v_{k-1} u_k^{x_k} v_k$ be a knapsack expression over the alphabet $\Sigma \cup \{t, t^{-1}\}$. We define the knapsack expression $\pi_\Sigma(e) = \pi_\Sigma(v_0) \pi_\Sigma(u_1)^{x_1} \cdots \pi_\Sigma(v_{k-1}) \pi_\Sigma(u_k)^{x_k} \pi_\Sigma(v_k)$ over the alphabet Σ. The assertion $e(x_1, \ldots, x_n) \in_H G$ is called a *G-constraint*. If e is a knapsack expression over the alphabet Σ, then $e \in_G A$ is called an A-constraint. Since G is knapsack semilinear relative to A, the set of solutions of an A-constraint is semilinear.

Lemma 5.2. *Let $u, v \in \mathsf{BR}(H)$ be well-behaved, u' (resp., v'') be a proper prefix of u (resp., v) and u'' (resp., v') be a proper suffix of u (resp., v). Let $e = e(z_1, \ldots, z_k)$ be a knapsack expression over the alphabet Σ. Then the set of all $(x, y, z_1, \ldots, z_k) \in \mathbb{N}^{k+2}$ such that the G-constraint*

$$u'' u^x u' e(z_1, \ldots, z_k)\, v' v^y v'' \in_H G \tag{7}$$

holds is semilinear and a semilinear representation can be effectively computed from $u, v, u', u'', v', v'', e$.

Proof. By cyclically rotating u and v we can assume that $u'' = v'' = \varepsilon$. Thus, we have to consider the set of solutions of the G-constraint $u^x u' e(z_1, \ldots, z_k)\, v' v^y \in_H G$. This equation holds (for certain values of $x, y, z_1, \ldots, z_k$) iff the word $u^x u' e(z_1, \ldots, z_k)\, v' v^y$ can be Britton-reduced to a word from Σ^* (this word must be $\pi_\Sigma(u^x u' e(z_1, \ldots, z_k)\, v' v^y)$). Since u^x and v^y are Britton-reduced for every $x, y \in \mathbb{N}$ we can apply Lemma 4.2.

Let S_u be the set of suffixes of u that start with $t^{\pm 1}$ and let P_v be the set of prefixes of v that end with $t^{\pm 1}$. We define $S_{u'}$ and $P_{v'}$ analogously. Then by Lemma 4.2 the following formula is equivalent to $u^x u' e(z_1, \ldots, z_n)\, v' v^y \in_H G$:

$$\begin{aligned}
&\pi_t(u^x u') = \pi_t(v' v^y)^{-1} \ \wedge \\
&\forall x' < x\, \forall y' < y\, \forall s \in S_u\, \forall p \in P_v\, \forall s' \in S_{u'}\, \forall p' \in P_{v'} : \\
&\qquad \pi_t(s\, u^{x'} u') = \pi_t(v' v^{y'} p)^{-1} \to \pi_\Sigma(s\, u^{x'} u' e\, v' v^{y'} p) \in_G A \ \wedge \\
&\qquad \pi_t(s\, u^{x'} u') = \pi_t(p')^{-1} \to \pi_\Sigma(s\, u^{x'} u' e\, p') \in_G A \ \wedge \\
&\qquad \pi_t(s') = \pi_t(v' v^{y'} p)^{-1} \to \pi_\Sigma(s' e\, v' v^{y'} p) \in_G A \ \wedge \\
&\qquad \pi_t(s') = \pi_t(p')^{-1} \to \pi_\Sigma(s' e\, p') \in_G A
\end{aligned}$$

Note that $\forall s \in S_u\, \forall p \in P_v\, \forall s' \in S_{u'}\, \forall p' \in P_{v'}$ can be written as a finite conjunction. By Lemma 2.2 the solution set of the equation $\pi_t(u^x u') = \pi_t(v' v^y)^{-1}$ (which is interpreted over the free monoid $\{t, t^{-1}\}^*$) is semilinear. To see this let $w = v^{-1}$ and $w' = (v')^{-1}$. Then $\pi_t(u^x u') = \pi_t(v' v^y)^{-1}$ is equivalent to the equation $\pi_t(u)^x \pi_t(u') = \pi_t(w)^y \pi_t(w')$. For the same reason, also the equations $\pi_t(s\, u^{x'} u') = \pi_t(v' v^{y'} p)^{-1}$ is equivalent to a semilinear constraint. The solution sets of the equations $\pi_t(s) = \pi_t(v' v^{y'} p)^{-1}$ and $\pi_t(s) = \pi_t(v' v^{y'} p)^{-1}$ are finite. Moreover, each of the A-constraints ($\pi_\Sigma(s\, u^{x'} u'\, e\, v' v^{y'} p) \in_G A$ etc.) is equivalent to a semilinear constraint because G is knapsack semilinear relative to A. Hence, the above formula is equivalent to a Presburger formula and therefore defines a semilinear set. ◻

Remark 5.3. There are variations of Lemma 5.2, where in the G-constraint (7), the subexpression $u'' u^x u'$ or $v' v^y v''$ (or both of them) is replaced by a single word from $\mathsf{BR}(H)$ (which does not contain an exponent variable). In all cases, the set of solutions of the G-constraint can be shown to be effectively semilinear using the arguments from the proof of Lemma 5.2.

Definition 5.4. We define a reduction relation on tuples over $\mathsf{BR}(H)$ of arbitrary length. A tuple $(u_1, \ldots, u_m)$ with $u_1, \ldots, u_m \in \mathsf{BR}(H)$ can be rewritten into

$$(u_1, \ldots, u_{i-1}, \pi_\Sigma(u_i), u_{i+1}, \ldots, u_{j-1}, \pi_\Sigma(u_j), u_{j+1}, \ldots, u_m)$$

if $1 \le i < j \le m$, $\pi_t(u_i) \ne \varepsilon \ne \pi_t(u_j)$, $u_{i+1} \cdots u_{j-1} \in \Sigma^*$ and $u_i u_{i+1} \cdots u_{j-1} u_j \in_H G$. Note that this implies

$$u_i u_{i+1} \cdots u_{j-1} u_j =_H \pi_\Sigma(u_i) u_{i+1} \cdots u_{j-1} \pi_\Sigma(u_j).$$

A concrete sequence of such rewrite steps leading to a tuple of words over Σ is a *G-reduction* of the initial tuple, and the initial tuple is called *G-reducible*. We also say that u_i and u_j are *matched in a G-reduction.*

A G-reduction of a tuple $(u_1, \ldots, u_m)$ can be seen as a witness for the fact that $u_1 \cdots u_m \in_H G$. On the other hand, $u_1 \cdots u_m \in_H G$ does not necessarily imply that $(u_1, \ldots, u_m)$ has a G-reduction. But we can show that $u_1 \cdots u_m \in_H G$ implies that $(u_1, \ldots, u_m)$ can refined (by factorizing the u_i) such that the resulting refined tuple has a G-reduction. Moreover, it is important that the length of the refined tuple only depends on m and not in the words $u_1, \ldots, u_m$.

We say that the tuple $(v_1, v_2, \ldots, v_n)$ is a *refinement* of the tuple $(u_1, u_2, \ldots, u_m)$ if there exist factorizations $u_i = u_{i,1} \cdots u_{i,k_i}$ in the free monoid $(\Sigma \cup \{t, t^{-1}\})^*$ such that $k_i = 1$ whenever $u_i \in \Sigma^*$ and $(v_1, v_2, \ldots, v_n) = (u_{1,1}, \ldots, u_{1,k_1}, \ldots, u_{m,1}, \ldots, u_{m,k_m})$.

Lemma 5.5. *Assume that $m \ge 2$ and $u_1, u_2, \ldots, u_m \in \mathrm{BR}(H)$. If $u_1 u_2 \cdots u_m \in_H G$, then there exists a G-reducible refinement of $(u_1, u_2, \ldots, u_m)$ that has length at most $4m$.*

Proof. Let $\overline{u} = (u_1, u_2, \ldots, u_m)$. Let us define $\gamma(\overline{u})$ as the number of pairs (i, j) with $1 \le i < j \le m$ such that $u_i u_{i+1} \cdots u_j$ is not Britton-reduced and $u_{i+1} \cdots u_{j-1} \in \Sigma^*$. Note that $\pi_t(u_i) \ne \varepsilon \ne \pi_t(u_j)$ for such a pair (i, j). Moreover, if we have two pairs (i, j) and (k, ℓ) of this form, then either $j \le k$ or $\ell \le i$. Let $\tau(\overline{u})$ be the number of i such that $\pi_t(u_i) \ne \varepsilon$.

We prove by induction over $\gamma(\overline{u}) + \tau(\overline{u})$ that there exists a G-reducible refinement of $\overline{u}$ of length at most $2\gamma(\overline{u}) + \tau(\overline{u}) + m \le 4m$.

The case $m = 2$ is trivial: either $\gamma(u_1, u_2) = \tau(u_1, u_2) = 0$ and $u_1, u_2 \in \Sigma^*$ or $\gamma(u_1, u_2) = 1$, $\tau(u_1, u_2) = 2$ in which case (u_1, u_2) must reduce in one step to $(\pi_\Sigma(u_1), \pi_\Sigma(u_2))$. If $m \ge 3$ then $u_1 u_2 \cdots u_m$ must contain a pin. Since every u_i is Britton-reduced, there must exist $i < j$ such that $u_i u_{i+1} \cdots u_j$ is not Britton-reduced and $u_{i+1} \cdots u_{j-1} \in \Sigma^*$. By Lemma 4.2 we can factorize u_i and u_j in $(\Sigma \cup \{t, t^{-1}\})^*$ as $u_i = u_i' r$ and $u_j = s u_j'$ such that $rs \in_H G$ and $u_i u_{i+1} \cdots u_{j-1} u_j =_H u_i' \pi_\Sigma(r) u_{i+1} \cdots u_{j-1} \pi_\Sigma(s) u_j'$ is Britton-reduced. Note that r and s must contain t or t^{-1}. Moreover, we can assume that either $u_i' = \varepsilon$ or u_i' ends with $t^{\pm 1}$ and, similarly, either $u_j' = \varepsilon$ or u_j' begins with $t^{\pm 1}$.

Case 1. u_i' and u_j' both contain $t^{\pm 1}$. Let

$$\overline{u}' = (u_1, \ldots, u_{i-1}, u_i', \pi_\Sigma(r), u_{i+1}, \ldots, u_{j-1}, \pi_\Sigma(s), u_j', u_{j+1}, \ldots, u_m).$$

We then have $\gamma(\overline{u}') < \gamma(\overline{u})$ since $u_i' \pi_\Sigma(r) u_{i+1} \cdots u_{j-1} \pi_\Sigma(s) u_j'$ is Britton-reduced and u_i' and u_j' both contain $t^{\pm 1}$. Moreover, $\tau(\overline{u}') = \tau(\overline{u})$. Hence, we can apply the induction hypothesis to the tuple $\overline{u}'$. It must have a G-reducible refinement of length at most

$$2(\gamma(\overline{u}) - 1) + \tau(\overline{u}) + m + 2 = 2\gamma(\overline{u}) + \tau(\overline{u}) + m.$$

In this refinement $\pi_\Sigma(r), \pi_\Sigma(s) \in \Sigma^*$ will not be factorized into more than one factor. We therefore can take the refinement of $\overline{u}'$ and replace $\pi_\Sigma(r)$ and $\pi_\Sigma(s)$ by r and s, respectively. This leads to a G-reducible of our original tuple $\overline{u}$ having length at most $2\gamma(\overline{u}) + \tau(\overline{u}) + m$.

Case 2. $u_i' = \varepsilon$ and u_j' begins with $t^{\pm 1}$. Let

$$\overline{u}' = (u_1, \ldots, u_{i-1}, \pi_\Sigma(u_i), u_{i+1}, \ldots, u_{j-1}, \pi_\Sigma(s), u_j', u_{j+1}, \ldots, u_m).$$

We have $\gamma(\overline{u}') \le \gamma(\overline{u})$ and $\tau(\overline{u}') < \tau(\overline{u})$. We can therefore apply the induction hypothesis to the tuple $\overline{u}'$ and obtain a G-reducible refinement of length at most $2\gamma(\overline{u}) + \tau(\overline{u}) - 1 + m + 1 = 2\gamma(\overline{u}) + \tau(\overline{u}) + m$. Replacing $\pi_\Sigma(u_i)$ by u_i and $\pi_\Sigma(s)$ by s in this refinement yields a G-reducible refinement of $\overline{u}$.

The remaining cases where (i) $u_j' = \varepsilon$ and u_i' ends with $t^{\pm 1}$ or (ii) $u_i' = u_j' = \varepsilon$ are analogous to case 2. □

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is based on ideas from [10]. Consider a knapsack expression

$$e(x_2, x_4, \ldots, x_m) = u_1 u_2^{x_2} u_3 u_4^{x_4} \cdots u_{m-1} u_m^{x_m} u_{m+1}$$

with m even (later it will convenient to have only variables with an even index). We can assume that all u_i are Britton reduced. Moreover, by Lemma 5.1, we can assume that every u_i with i even is well-behaved and non-empty (otherwise we can remove $u_i^{x_i}$).

In the following we describe an algorithm that computes a semilinear representation for the set of all H-solutions of $e = 1$. The algorithm transforms logical statements into equivalent logical statements (we do not have to define the precise logical language; the meaning of the statements should be always clear). Every statement contains the free variables $x_2, x_4, \ldots, x_m$ from our knapsack expression and equivalence of two statements means that for all values in $\mathbb{N}$ that $x_2, x_4, \ldots, x_m$ can take, the two statements yield the same truth value. We start with the statement $e(x_2, x_4, \ldots, x_m) =_H 1$ and end with a Presburger formula. In each step we transform the current statement Φ into an equivalent disjunction $\bigvee_{i=1}^n \Phi_i$. In the next step, we will then concentrate on a single Φ_i.

Let $N_\Sigma \subseteq [1, m+1]$ be the set of indices such that $u_i \in \Sigma^*$ and let $N_t = [1, m+1] \setminus N_\Sigma$ be the set of indices such that $\pi_t(u_i) \ne \varepsilon$. Moreover, let us define $w_i = u_i$ for i odd and $w_i = u_i^{x_i}$ for i even.

By Lemma 4.1, $e =_H 1$ is equivalent to $e \in_H G \wedge \pi_\Sigma(e) =_G 1$. Since G is knapsack semilinear, the set of all solutions of $\pi_\Sigma(e) =_G 1$ is semilinear. It therefore suffices to show that the set of all $(x_2, x_4, \ldots, x_m) \in \mathbb{N}^{m/2}$ with $e(x_2, x_4 \ldots, x_m) \in_H G$ is semilinear. Here, we will use the assumption that G is knapsack semilinear relative to A.

Step 1: Applying Lemma 5.5. We construct a disjunction Ψ from the knapsack expression e using Lemma 5.5. More precisely, Ψ is obtained by taking the disjunction over the following choices:

(i) symbolic factorizations $w_i = y_{i,1} \cdots y_{i,k_i}$ in $(\Sigma \cup \{t, t^{-1}\})^*$ for all $i \in [1, m+1]$. Here the $y_{i,j}$ are existentially quantified variables that take values in $\mathrm{BR}(H)$. Later, these variables will be eliminated. The k_i must satisfy $k_i \ge 1$ for all i, $k_i = 1$ for all $i \in N_\Sigma$, and $\sum_{1 \le i \le m+1} k_i \le 4(m+1)$.

(ii) a G-reduction (according to Definition 5.4) of the tuple

$$(y_{1,1} \cdots y_{1,k_1}, \ldots, y_{m+1,1} \cdots y_{m+1,k_{m+1}}).$$

For every specific choice in (i) and (ii) we write down the conjunction of the following formulas:

- the equation $w_i = y_{i,1} \cdots y_{i,k_i}$ from (i) (every variable $y_{i,j}$ is existentially quantified) and
- all G-constraints that result from G-reduction steps in the G-reduction from (ii) (this will made precise in Step 2 below).

The formula Ψ is the disjunction of the above existentially quantified conjunctions, taken over all possible choices in (i) and (ii). This formula is equivalent to the G-constraint $e \in_H G$.

Step 2: Eliminating the equations $w_i = y_{i,1} \cdots y_{i,k_i}$. For an odd i (i.e., $w_i = u_i$) we can eliminate this equation by taking a disjunction over all concrete factorizations $u_i = u_{i,1} \cdots u_{i,k_i}$ and then replace the equation $w_i = y_{i,1} \cdots y_{i,k_i}$ by the conjunction of all equations $y_{i,j} = u_{i,j}$ for $1 \le j \le k_i$. For an even i (i.e., $w_i = u_i^{x_i}$) we can eliminate the equation $w_i = y_{i,1} \cdots y_{i,k_i}$ by taking a disjunction over all symbolic factorizations of $u_i^{x_i}$ into k_i factors. A specific factorization leads to a formula

$$\bigwedge_{j=1}^{k_i} y_{i,j} = u''_{i,j} u_i^{x_{i,j}} u'_{i,j+1} \ \wedge\ x_i = c_i + \sum_{j=1}^{k_i} x_{i,j}. \tag{8}$$

Here, every $u'_{i,j}$ ($2 \le j \le k_i$) is a proper prefix of u_i and every $u''_{i,j}$ ($2 \le j \le k_i$) is a proper suffix of u_i such that either $u_i = u'_{i,j} u''_{i,j}$ or $u'_{i,j} = u''_{i,j} = \varepsilon$ for all $2 \le j \le k_i$. We set $u'_{i,k_i+1} = u''_{i,1} = \varepsilon$ in the above formula. Moreover, c_i is the number of $2 \le j \le k_i$ for which $u'_{i,j} \neq \varepsilon \neq u''_{i,j}$ holds. The disjunction has to be taken over all choices for the $u'_{i,j}$ and $u''_{i,j}$. The $x_{i,j}$ are new existentially quantified exponent variables.

We also take for every $x_{i,j}$ a disjunction over the two choices $x_{i,j} = 0$ and $x_{i,j} > 0$. If $x_{i,j} = 0$, then we replace $x_{i,j}$ in (8) by 0. This yields the equation $y_{i,j} = u''_{i,j} u'_{i,j+1}$. If $x_{i,j} > 0$, then we add this constraint to (8). After this step, it is determined whether a $y_{i,j}$ contains t or t^{-1} (for i even as well as for i odd). Those $y_{i,j}$ must be matched by G-reduction steps in the G-reduction from Step 1. In fact, the disjunction in Step 1 is taken over all such matchings.

Step 3: Eliminating G-constraints. Assume that $y_{i,j}$ and $y_{k,\ell}$ are matched in the G-reduction from Step 1. W.l.o.g. assume that $i < k$ or $i = k$ and $j < \ell$, i.e., (i,j) is lexicographically before (k,ℓ). Then our formula contains the G-constraint

$$y_{i,j} \Bigg(\prod_{(i,j) \prec (p,q) \prec (k,\ell)} \pi_\Sigma(y_{p,q}) \Bigg) y_{k,\ell} \in_H G,$$

where $\prec$ is the strict lexicographic order on pairs of natural numbers. In this constraint, we can replace every $y_{a,b}$ with a even by $u''_{i,j} u_i^{x_{i,j}} u'_{i,j+1}$ (or $u''_{a,b} u'_{a,b+1}$ in case $x_{i,j} = 0$), whereas every $y_{a,b}$ with a odd can be replaced by the concrete word $u_{a,b}$. This leads either to a G-constraint of the form (7) (if $y_{i,j}$ and $y_{k,\ell}$ both contain an exponent variable) or to a simpler G-constraint, where $y_{i,j}$ or $y_{k,\ell}$ is a concrete word. In the former case the set of solutions of the G-constraint is semilinear by Lemma 5.2. The latter case is covered by Remark 5.3. In this way we finally obtain a Presburger formula equivalent to $e(x_2, x_4 \ldots, x_m) \in_H G$. This concludes the proof. □

For a subset $S \subseteq G$ of the group G one defines the *centralizer* of S as the subgroup $C(S) = \{h \in G \mid gh = hg \text{ for all } g \in S\} \le G$. The HNN-extension $H = \langle G, t \mid t^{-1}at = a \ (a \in C(S)) \rangle$ is an *extension of the centralizer* $C(S)$. Extensions of centralizers were first studied in [29] in the context of exponential groups.

Theorem 5.6. *If G is knapsack semilinear and H is an extension of a centralizer $C(S)$ with S finite, then also H is knapsack semilinear.*

Proof. We show that G is knapsack semilinear relative to $C(S)$. Let $e = e(x_1, \ldots, x_n)$ be a knapsack expression. Then $e \in_G C(S)$ is equivalent to $\bigwedge_{a \in S} ea =_G ae$. Note that $ea =_G ae$ is equivalent to $eae^{-1}a^{-1} =_G 1$ and $eae^{-1}a^{-1}$ is an exponent expression. Since G is knapsack semilinear and semilinear sets are closed under finite intersections, the set of solutions of $\bigwedge_{a \in S} ea = ae$ is semilinear. □

Remark 5.7. Theorem 1.2 can be generalized to multiple HNN-extensions $H = \langle G, t_1, \ldots, t_n \mid t_i^{-1} a t_i = a \ (a \in A_i, 1 \le i \le n) \rangle$. If G is knapsack semilinear relative to $\{1, A_1, \ldots, A_n\}$ then H is knapsack semilinear.

6 APPLICATION TO HYPERBOLIC GROUPS

In this section we show that hyperbolic groups are knapsack semilinear relative to quasiconvex subgroups. We start with the definition of hyperbolic groups.

6.1 Cayley-graphs and hyperbolic groups

Let G be a f.g. group with the finite symmetric generating set Σ and let $h : \Sigma^* \to G$ be the evaluation morphism. The *Cayley-graph* of G with respect to Σ is the undirected graph $\Gamma = \Gamma(G)$ with node set G and all edges (g, ga) for $g \in G$ and $a \in \Sigma$. We view Γ as a geodesic metric space, where every edge (g, ga) is identified with a unit-length interval. It is convenient to label the directed edge from g to ga with the generator a. The distance between two points p, q is denoted with $d_\Gamma(p, q)$.

Paths can be defined in a very general way for metric spaces, but we only need paths that are induced by words over Σ. Given a word $w = a_1 a_2 \cdots a_n$ (with $a_i \in \Sigma$), one obtains a unique path $P[w] : [0, n] \to \Gamma$, which is a continuous mapping from the real interval $[0, n]$ to Γ. It maps the subinterval $[i, i+1] \subseteq [0, n]$ isometrically onto the edge $(h(a_1 \cdots a_i), h(a_1 \cdots a_{i+1}))$ of Γ. The path $P[w]$ starts in 1 and ends in $h(w)$ (the group element represented by w). We also say that $P[w]$ is the unique path that starts in 1 and is labelled with the word w. More generally, for $g \in G$ we denote with $g \cdot P[w]$ the path that starts in g and is labelled with w. When writing $u \cdot P[w]$ for a word $u \in \Sigma^*$, we mean the path $h(u) \cdot P[w]$. A path $P : [0, n] \to \Gamma$ of the above form is *geodesic* if $d_\Gamma(P(0), P(n)) = n$ and it is a (λ, ϵ)-*quasigeodesic* if for all points $p = P(a)$ and $q = P(b)$ we have $|a - b| \le \lambda \cdot d_\Gamma(p, q) + \varepsilon$.

A word $w \in \Sigma^*$ is geodesic if the path $P[w]$ is geodesic, which means that there is no shorter word representing the same group element from G. Similarly, we define the notion of (λ, ϵ)-quasigeodesic words. A set of words $L \subseteq \Sigma^*$ is called $((\lambda, \epsilon)$-quasi)geodesic, if every $w \in L$ is $((\lambda, \epsilon)$-quasi)geodesic.

A *geodesic triangle* in Γ is a triangle whose three sides are geodesic paths. A geodesic triangle is δ-*slim* for $\delta \ge 0$, if every point on each side of the triangle has distance at most δ to a point that belongs to one of the two other sides. The group G is called δ-*hyperbolic*, if every geodesic triangle is δ-slim. Finally, G is hyperbolic, if it is δ-hyperbolic for some $\delta \ge 0$. This property is independent of the chosen generating set Σ. Finitely generated free groups are for instance 0-hyperbolic. The word problem for every hyperbolic group can be decided in real time [18].

6.2 Asynchronous biautomatic structures

Let G be a f.g. group with the finite symmetric generating set Σ and let $h : \Sigma^* \to G$ be the evaluation morphism. An *asynchronous biautomatic structure* for G consists of a regular language $L \subseteq \Sigma^*$ such that $G = h(L)$ and for all $a, b \in \Sigma \cup \{\varepsilon\}$ with $|ab| \leq 1$ the relation $\{(u, v) \in L \times L \mid aub =_G v\}$ is rational; see also [6, 30]. If we only require rationality of $\{(u, v) \in L \times L \mid ua =_G v\}$ for $a \in \Sigma \cup \{\varepsilon\}$, then L is an asynchronous automatic structure for G. A f.g. group G is called asynchronously (bi)automatic if it has an asynchronous (bi)automatic structure. We need the following lemma.

Lemma 6.1. *Let L be an asynchronous biautomatic structure for G, let L_1 and L_2 be regular subsets of L and let $v_1, v_2 \in \Sigma^*$. Then the relation $\{(u_1, u_2) \in L_1 \times L_2 \mid v_1u_1 =_G u_2v_2\}$ is rational. Moreover, a finite state transducer for this relation can be effectively computed from the words v_1, v_2 and finite automata for L_1 and L_2.*

Proof. It suffices to show that the relation

$$R := \{(u_1, u_2) \in L \times L \mid v_1u_1 =_G u_2v_2\}$$

is rational. The corresponding finite state transducer can in addition simulate the automaton for L_1 (resp., L_2) on the first (resp., second) tape. Rationality of the relation R can be shown by induction on $|v_1| + |v_2|$. The case $v_1 = v_2 = \varepsilon$ is clear. Assume w.l.o.g. that $v_1 \neq \varepsilon$ and let $v_1 = v_1'a$ with $a \in \Sigma$. By induction, the relation $R_1 = \{(u_1', u_2) \in L \times L \mid v_1'u_1' =_G u_2v_2\}$ is rational. Moreover, the relation $R_2 = \{(u_1, u_1') \in L \times L \mid au_1 =_G u_1'\}$ is rational as well. Finally, we have $R = R_2 \circ R_1$, where $\circ$ is relational composition. The lemma follows since the class of rational relations is closed under relational composition [35]. ◻

We also need the following result from [19]:

Lemma 6.2. *Let G be a hyperbolic group and let Σ be a finite symmetric generating set for G. Let λ and ϵ be fixed constants. Then the set of all (λ, ϵ)-quasigeodesic words over the alphabet Σ is an asynchronous biautomatic structure for G.*

In [19] it is only stated that the set of all (λ, ϵ)-quasigeodesic words is an asynchronous automatic structure for G. But since for every (λ, ϵ)-quasigeodesic word $w \in \Sigma^*$ also w^{-1} is (λ, ϵ)-quasigeodesic, it follows easily that the set of all (λ, ϵ)-quasigeodesic words is an asynchronous biautomatic structure for G. Lemma 6.1 yields the following lemma.

Lemma 6.3. *Assume that λ and ϵ are fixed constants, $L_1, L_2 \subseteq \Sigma^*$ are (λ, ϵ)-quasigeodesic regular languages, and $v_1, v_2 \in \Sigma^*$. Then the relation $\{(u_1, u_2) \in L_1 \times L_2 \mid v_1u_1 =_G u_2v_2\}$ is rational. Moreover, a finite state transducer for this relation can be effectively computed from the words v_1, v_2 and finite automata for L_1 and L_2.*

6.3 Parikh images in hyperbolic groups

Let us fix a hyperbolic group G with the finite symmetric generating set Σ for the rest of the section. We fix an arbitrary enumeration $a_1, \ldots, a_k$ of the alphabet Σ in order to make Parikh images well-defined. Recall that the semilinear sets are exactly the Parikh images of regular languages; see Theorem 2.1. Together with Lemma 6.3 we obtain the next result.

Lemma 6.4. *Assume that λ, ϵ are fixed constants, $L_1, L_2 \subseteq \Sigma^*$ are (λ, ϵ)-quasigeodesic regular languages, and $v_1, v_2 \in \Sigma^*$. Then the set*

$$\{(P(u_1), P(u_2)) \in \mathbb{N}^{2k} \mid u_1 \in L_1, u_2 \in L_2, v_1u_1 =_G u_2v_2\} \quad (9)$$

is semilinear and a semilinear representation can be effectively computed from the words v_1, v_2 and finite automata for L_1 and L_2.

Proof. Let $\Sigma' = \{a' \mid a \in \Sigma\}$ be a disjoint copy of the alphabet Σ. By Lemma 6.3 there is a finite state transducer $\mathcal{T}$ for the relation $\{(u_1, u_2) \in L_1 \times L_2 \mid v_1u_1 =_G u_2v_2\}$. From $\mathcal{T}$ we obtain a finite automaton $\mathcal{A}$ over the alphabet $\Sigma \cup \Sigma'$ by replacing every transition (p, a, ε, q) by (p, a, q) and every transition (p, ε, a, q) by (p, a', q). For the alphabet $\Sigma \cup \Sigma'$ we take the enumeration $a_1, \ldots, a_k, a_1', \ldots, a_k'$. With this enumeration, the set (9) is the Parikh image of the language $L(\mathcal{A})$. Hence, the lemma follows from Theorem 2.1. ◻

6.4 The main result

We now come to the main technical result of this section.

Theorem 6.5. *Let G be a hyperbolic group with the finite symmetric generating set $\Sigma = \{a_1, \ldots, a_k\}$. Fix constants ϵ, λ. For $1 \leq i \leq n$ let $L_i \subseteq \Sigma^*$ be a regular (λ, ϵ)-quasigeodesic language. Then, the set*

$$\{(P(w_1), \ldots, P(w_n)) \in \mathbb{N}^{nk} \mid w_i \in L_i \ (1 \leq i \leq n), w_1 \cdots w_n =_G 1\}$$

is semilinear and a semilinear representation of this set can be computed from finite automata for $L_1, \ldots, L_n$.

Proof. Let G be δ-hyperbolic. For $1 \leq i \leq n$ let $L_i \subseteq \Sigma^*$ be a regular (λ, ϵ)-quasigeodesic language. We want to show that the subset of $\mathbb{N}^{nk}$ is semilinear. For this, we prove a slightly more general statement: For words $v_1, \ldots, v_n \in \Sigma^*$ we consider the set

$$\{(P(w_1), \ldots, P(w_n)) \in \mathbb{N}^{nk} \mid w_i \in L_i \ (1 \leq i \leq n), \\ w_1v_1 \cdots w_nv_n =_G 1\}.$$

By induction over n we show that this set is semilinear. For the case $n = 2$ we can directly use Lemma 6.4. This also covers the case $n = 1$ since we can take $L_2 = \{1\}$.

Now assume that $n \geq 3$. We can assume that the words v_i are geodesic. We can also assume that there is a single finite automaton $\mathcal{A}$ with state set Q such that for every L_i there are subsets $S_i, T_i \subseteq Q$ such that L_i is the set of all words that label a path from a state in S_i to a state in T_i. Let us denote for $p, q \in Q$ with $L_{p,q}$ the set of all finite words that label a path from p to q in the automaton $\mathcal{A}$. We can assume that all these languages are (λ, ϵ)-quasigeodesic. Note that $L_i = \bigcup_{p \in S_i, q \in T_i} L_{p,q}$. Since the semilinear sets are effectively closed under union, it suffices to show for states $p_i, q_i \in Q$ $(1 \leq i \leq n)$ that the following set is semilinear:

$$\{(P(w_1), \ldots, P(w_n)) \in \mathbb{N}^{nk} \mid w_i \in L_{p_i,q_i} \ (1 \leq i \leq n), \\ w_1v_1 \cdots w_nv_n =_G 1\}$$

We denote this set with $P(p_1, q_1, v_1, \ldots, p_n, q_n, v_n)$ and construct a Presburger formula with free variables $x_{i,j}$ $(1 \leq i \leq n, 1 \leq j \leq k)$ for it. The variables $x_{i,j}$ with $1 \leq j \leq k$ encode the Parikh image of the words from L_{p_i,q_i}. Let us write $\bar{x}_i = (x_{i,j})_{1 \leq j \leq k}$.

Consider $(w_1, \ldots, w_n) \in \prod_{i=1}^{n} L_{p_i,q_i}$ with $w_1v_1 \cdots w_nv_n =_G 1$ and the corresponding $2n$-gon that is defined by the (λ, ϵ)-quasigeodesic paths $P_i = (w_1v_1 \cdots w_{i-1}v_{i-1}) \cdot P[w_i]$ and the geodesic paths $Q_i = (w_1v_1 \cdots w_i) \cdot P[v_i]$, see Figure 1 for the case $n = 3$.

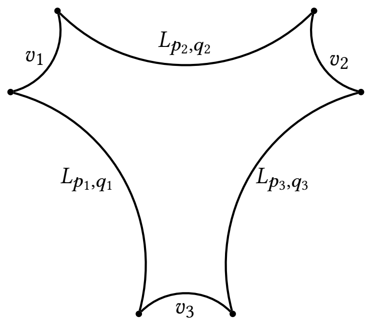

Figure 1: The $2n$-gon for $n = 3$ from the proof of Theorem 6.5

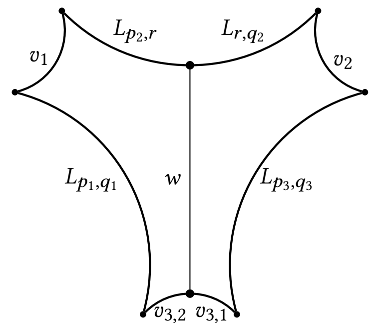

Figure 2: A typical case from the proof of Theorem 6.5

Since all paths P_i and Q_i are (λ, ϵ)-quasigeodesic, we can apply [28, Lemma 6.4]: every side of the $2n$-gon is contained in the κ-neighborhoods of the other sides, where $\kappa = \xi + \xi \log(2n)$ for a constant ξ that only depends on the constants $\delta, \lambda, \varepsilon$. This allows to cut the $2n$-gon into several $2m$-gons (for values $m < n$) along paths of length at most κ. To each of these smaller $2m$-gons we can then apply the induction hypothesis. The end points of the paths, along which we cut the $2n$-gon, have to be carefully chose in order to ensure that the resulting $2m$-gons satisfy $m > n$. This leads to several cases. A systematic consideration of all cases can be found in [9]; it follows the proof of the knapsack semilinearity of hyperbolic groups from [23]. Here, we only want to consider one typical case: Assume there is a path P of length at most κ connecting a point a on P_2 ($1 \le i \le n$) with a point on Q_i with $3 \le i \le n$. Assume that the path P is labelled with the word $w \in \Sigma^*$. The situation is shown in Figure 2 for $n = i = 3$. Let T be the set of all tuples $(r, v_{i,1}, v_{i,2}, w)$ such that $r \in Q$, $v_i = v_{i,1}v_{i,2}$, and $w \in \Sigma^*$ is of length at most κ. By induction, the following two sets are semilinear for every tuple $t = (r, v_{i,1}, v_{i,2}, w) \in T$:

$$\begin{aligned} S_{t,1} &= P(p_1, q_1, v_1, p_2, r, wv_{i,2}, p_{i+1}, q_{i+1}, v_{i+1}, \dots, p_n, q_n, v_n) \\ S_{t,2} &= P(r, q_2, v_2, p_3, q_3, v_3, \dots, p_i, q_i, v_{i,1}w^{-1}) \end{aligned}$$

Intuitively, $S_{t,1}$ corresponds to the $2(n-1)$-gon (when $wv_{i,2}$ is viewed as a single side) on the left of the w-labelled edge in Figure 2, whereas $S_{t,2}$ corresponds to the $2(n-1)$-gon on the right of the w-labelled edge. We then define the formula

$$\begin{aligned} A_1 = \bigvee_{t \in T} \exists \bar{y}_2, \bar{z}_2 : (\bar{x}_1, \bar{y}_2, \bar{x}_{i+1}, \dots, \bar{x}_n) \in S_{t,1} \,\wedge \\ (\bar{z}_2, \bar{x}_3, \dots, \bar{x}_i) \in S_{t,2} \;\wedge\; \bar{x}_2 = \bar{y}_2 + \bar{z}_2. \end{aligned}$$

Here $\bar{y}_2$ and $\bar{z}_2$ are k-tuples of new variables.

In [9], five more cases are considered, which lead to similar formulas $A_2, \dots, A_6$. Finally, a tuple $(\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{N}^{nk}$ belongs to the set $P(p_1, q_1, v_1, \dots, p_n, q_n, v_n)$ iff $\bigvee_{1 \le i \le 6} A_i$ holds. This yields a Presburger formula for $P(p_1, q_1, v_1, \dots, p_n, q_n, v_n)$. □

Let us derive some corollaries from Theorem 6.5.

Theorem 6.6. *Let G be hyperbolic and let $S \subseteq \Sigma^*$ be a regular geodesic language. Then G is knapsack semilinear relative to $h(S)$, where $h : \Sigma^* \to G$ is the evaluation morphism.*

Proof. Consider the knapsack expression

$$e = v_0 u_1^{x_1} v_1 u_1^{x_1} \cdots v_{n-1} u_n^{x_n} v_n$$

over the alphabet Σ. We want to find a semilinear representation for the set of all G-solutions of $e = 1$. In a first step, one modifies e in such a way that for every power $u_i^{x_i}$ that appears in e the language u_i^* is (λ, ϵ)-quasigeodesic for fixed constants λ, ϵ that only depend on the group G. This is done exactly in the same way as in the proof of [23, Proposition 8.4]. Clearly, we can also assume that every u_i is non-empty and every v_i is geodesic. Moreover, since S is regular and geodesic, it is easy to see that also S^{-1} is regular and geodesic.

Let $r = 2(n+1)$ and define the tuple of languages

$$(L_1, \dots, L_r) = (\{v_0\}, u_1^*, \{v_1\}, \dots, u_n^*, \{v_n\}, S^{-1}).$$

All the languages L_i are regular and (λ, ϵ)-quasigeodesic. By Theorem 6.5, the set

$$\{(P(w_1), \dots, P(w_r)) \in \mathbb{N}^{rk} \mid w_i \in L_i \ (1 \le i \le r), w_1 \cdots w_r =_G 1\}$$

is semilinear and a semilinear representation can be computed. Applying a projection yields a semilinear representation of the set

$$\begin{aligned} \{(P(w_1), \dots, P(w_n)) \in \mathbb{N}^{nk} \mid w_i \in u_i^* \text{ for } 1 \le i \le n, \\ \exists w \in S : v_0 w_1 v_1 \cdots w_n v_n =_G w\}. \end{aligned}$$

Choose for every u_i a symbol $a_{j_i} \in \Sigma$ such that $\ell_i := |u_i|_{a_{j_i}} > 0$ (recall that $u_i \neq \varepsilon$). Then we project every $P(w_i)$ in the above set to the j_i-th coordinate. The resulting projection is

$$\{(\ell_1 \cdot x_1, \dots, \ell_n \cdot x_n) \in \mathbb{N}^n \mid \exists w \in S : v_0 u_1^{x_1} \cdots v_{n-1} u_n^{x_n} v_n =_G w\}.$$

The semilinearity of this set easily implies the semilinearity of the set $\{(x_1, \dots, x_n) \in \mathbb{N}^n \mid \exists w \in S : v_0 u_1^{x_1} v_1 \cdots u_n^{x_n} v_n =_G w\}$. □

A subset $A \subseteq G$ is called *quasiconvex* if there exists a constant $\kappa \ge 0$ such that every point on a geodesic path from 1 to some $g \in A$ has distance at most κ from A. The following result can be found in [13] (h denotes the evaluation morphism):

Lemma 6.7. *A subset $A \subseteq G$ is quasiconvex if and only if the language of all geodesic words in $h^{-1}(A)$ is regular.*

Theorem 6.6 and Lemma 6.7 imply Theorem 1.3 from the introduction. Finally, Theorems 1.2 and 1.3 yield the following result:

Corollary 6.8. *If A is a quasiconvex subgroup of the hyperbolic group G then $\langle G, t \mid t^{-1}at = a \ (a \in A)\rangle$ is knapsack semilinear.*

Acknowledgement. Both authors were supported by the DFG research project LO 748/12-1.

REFERENCES

[1] László Babai, Robert Beals, Jin-Yi Cai, Gábor Ivanyos, and Eugene M. Luks. 1996. Multiplicative Equations over Commuting Matrices. In *Proceedings of SODA 1996*. ACM/SIAM, 498–507.

[2] Pascal Bergsträßer, Moses Ganardi, and Georg Zetzsche. 2021. A Characterization of Wreath Products Where Knapsack Is Decidable. In *Proceeding of the 38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021 (LIPIcs, Vol. 187)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 11:1–11:17. https://doi.org/10.4230/LIPIcs.STACS.2021.11

[3] Agnieszka Bier and Oleg Bogopolski. 2021. Exponential equations in acylindrically hyperbolic groups. arXiv:2106.11385

[4] Oleg Bogopolski and Aleksander Ivanov. 2021. Notes about decidability of exponential equations. arXiv:2105.06842

[5] Fedor A. Dudkin and Alexander V. Treyer. 2018. Knapsack problem for Baumslag–Solitar groups. *Siberian Journal of Pure and Applied Mathematics* 18, 4 (2018), 43–55. https://doi.org/10.33048/pam.2018.18.404

[6] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. 1992. *Word Processing in Groups*. Jones and Bartlett, Boston.

[7] David B. A. Epstein and Derek F. Holt. 2006. The Linearity of the Conjugacy Problem in Word-hyperbolic Groups. *International Journal of Algebra and Computation* 16, 2 (2006), 287–306.

[8] Michael Figelius, Moses Ganardi, Markus Lohrey, and Georg Zetzsche. 2020. The Complexity of Knapsack Problems in Wreath Products. In *Proceedings of the 47th International Colloquium on Automata, Languages, and Programming, ICALP 2020 (LIPIcs, Vol. 168)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 126:1–126:18.

[9] Michael Figelius and Markus Lohrey. 2022. Exponent equations in HNN-extensions. arXiv:2202.04038

[10] Michael Figelius, Markus Lohrey, and Georg Zetzsche. 2022. Closure properties of knapsack semilinear groups. *Journal of Algebra* 589, 1 (2022), 437–482. https://doi.org/10.1016/j.jalgebra.2021.08.016

[11] Elizaveta Frenkel, Andrey Nikolaev, and Alexander Ushakov. 2015. Knapsack problems in products of groups. *Journal of Symbolic Computation* 74 (2015), 96–108. DOI: doi:10.1016/j.jsc.2015.05.006.

[12] Moses Ganardi, Daniel König, Markus Lohrey, and Georg Zetzsche. 2018. Knapsack Problems for Wreath Products. In *Proceedings of 35th Symposium on Theoretical Aspects of Computer Science, STACS 2018 (LIPIcs, Vol. 96)*. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 32:1–32:13.

[13] Stephen M. Gersten and Hamish B. Short. 1991. Rational Subgroups of Biautomatic Groups. *Annals of Mathematics* 134, 1 (1991), 125–158. http://www.jstor.org/stable/2944334

[14] Mikhael Gromov. 1987. Hyperbolic groups. In *Essays in Group Theory (MSRI Publ., 8)*, S. M. Gersten (Ed.). Springer, 75–263.

[15] Christoph Haase. 2018. A Survival Guide to Presburger Arithmetic. *ACM SIGLOG News* 5, 3 (2018), 67–82. https://doi.org/10.1145/3242953.3242964

[16] Niko Haubold and Markus Lohrey. 2011. Compressed word problems in HNN-extensions and amalgamated products. *Theory of Computing Systems* 49, 2 (2011), 283–305.

[17] Graham Higman, Bernhard H. Neumann, and Hanna Neumann. 1949. Embedding theorems for groups. *Journal of the London Mathematical Society. Second Series* 24 (1949), 247–254.

[18] Derek Holt. 2000. Word-hyperbolic groups have real-time word problem. *International Journal of Algebra and Computation* 10 (2000), 221–228.

[19] Derek F. Holt and Sarah Rees. 2003. Regularity of quasigeodesics in a hyperbolic group. *International Journal of Algebra and Computation* 13, 05 (2003), 585–596. https://doi.org/10.1142/S0218196703001560

[20] John E. Hopcroft and Jeffrey D. Ullman. 1979. *Introduction to Automata Theory, Languages and Computation*. Addison–Wesley, Reading, MA.

[21] Olga Kharlampovich, Alexei Miasnikov, and Pascal Weil. 2017. Stallings graphs for quasi-convex subgroups. *Journal of Algebra* 488 (2017), 442–483. https://doi.org/10.1016/j.jalgebra.2017.05.037

[22] Daniel König, Markus Lohrey, and Georg Zetzsche. 2016. Knapsack and subset sum problems in nilpotent, polycyclic, and co-context-free groups. In *Algebra and Computer Science*. Contemporary Mathematics, Vol. 677. American Mathematical Society, 138–153.

[23] Markus Lohrey. 2019. Knapsack in hyperbolic groups. *Journal of Algebra* 545 (2019), 390–415. https://doi.org/10.1016/j.jalgebra.2019.04.008

[24] Markus Lohrey and Georg Zetzsche. 2018. Knapsack in Graph Groups. *Theory of Computing Systems* 62, 1 (2018), 192–246.

[25] Markus Lohrey and Georg Zetzsche. 2020. Knapsack and the Power Word Problem in Solvable Baumslag-Solitar Groups. In *Proceedings of the 45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020 (LIPIcs, Vol. 170)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 67:1–67:15.

[26] Roger C. Lyndon and Paul E. Schupp. 1977. *Combinatorial Group Theory*. Springer, NewYork Berlin Heidelberg.

[27] Alexei Mishchenko and Alexander Treier. 2017. Knapsack problem for nilpotent groups. *Groups Complexity Cryptology* 9, 1 (2017), 87–98.

[28] Alexei Myasnikov, Andrey Nikolaev, and Alexander Ushakov. 2015. Knapsack Problems in Groups. *Math. Comp.* 84 (2015), 987–1016.

[29] Alexei G. Myasnikov and Vladimir N. Remeslennikov. 1996. Exponential Groups 2: Extensions of Centralizers and Tensor Completion of CSA-Groups. *Int. J. Algebra Comput.* 6, 6 (1996), 687–712. https://doi.org/10.1142/S0218196796000398

[30] Walter D. Neumann and Michael Shapiro. 1994. Automatic structures and boundaries for graphs of groups. *International Journal of Algebra and Computation* 04, 04 (1994), 591–616. https://doi.org/10.1142/S0218196794000178

[31] Alexander Yu. Ol′shanskiĭ. 1992. Almost every group is hyperbolic. *International Journal of Algebra and Computation* 2, 1 (1992), 1–17.

[32] Rohit Parikh. 1966. On Context-Free Languages. *Journal of the ACM* 13, 4 (1966), 570–581. https://doi.org/10.1145/321356.321364

[33] Eliyahu Rips. 1982. Subgroups of small cancellation groups. *Bulletin of the London Mathematical Society* 14 (1982), 45–47.

[34] J. J. Rotman. 1995. *An Introduction to the Theory of Groups (4th edition)*. Springer, NewYork Berlin Heidelberg.

[35] David Simplot and Alain Terlutte. 2000. Closure under union and composition of iterated rational transductions. *RAIRO - Theoretical Informatics and Applications - Informatique Théorique et Applications* 34, 3 (2000), 183–212. http://www.numdam.org/item/ITA_2000__34_3_183_0/

[36] John Stillwell. 1993. *Classical Topology and Combinatorial Group Theory (2nd edition)*. Springer, NewYork Berlin Heidelberg.

[37] Daniel T. Wise. 2009. Research announcement: the structure of groups with a quasiconvex hierarchy. *Electronic Research Announcements in Mathematical Sciences* 16 (2009), 44–55.