Membership problems in infinite groups

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Abstract. We review results for various kinds of membership problems (subgroup membership, submonoid membership, rational subset membership, knapsack problem) in infinite groups.

Keywords: group theory · membership testing · rational sets.

1 Algorithmic membership problems in groups

Short historic outline. The investigation of membership problems in algorithmic group theory can be traced back to a paper of Markov from 1947 [47]. Markov proved that it is undecidable whether a given matrix A from the group $\mathsf{SL}_4(\mathbb{Z})$ of 4-dimensional integer matrices of determinant 1 can be written as a product (of arbitrary length) of other given matrices $A_1, \ldots, A_k \in \mathsf{SL}_4(\mathbb{Z})$. In the terminology that we introduce below, Markov showed that the *submonoid membership problem* for the matrix group $\mathsf{SL}_4(\mathbb{Z})$ is undecidable. Markov's work initiated extensive research on various algorithmic problems in low-dimensional matrix (semi)groups; see [5,16,37,51] for more recent work. Moreover, he introduced membership problems to group theory.

In general, a membership problem for a group G asks whether a given element $g \in G$ belongs to a given subset $S \subseteq G$. In order to get a well-defined decision problem, one has to restrict the input set S to a class of subsets having finitary representations. In Markov's case S is the submonoid generated by given matrices $A_1, \ldots, A_k \in \mathsf{SL}_4(\mathbb{Z})$. From a group theoretic perspective, it is also natural to consider the subgroup generated by the given matrices $A_1, \ldots, A_k \in \mathsf{SL}_4(\mathbb{Z})$. This leads to the *subgroup membership problem* for $\mathsf{SL}_4(\mathbb{Z})$, which is still undecidable by a result of Mihaĭlova [48]. Actually, Mihaĭlova proved that the subgroup membership problem is undecidable for the direct product of two free groups of rank 2, which is a subgroup of $\mathsf{SL}_4(\mathbb{Z})$. Following the work of Mihaĭlova, the subgroup membership problem has been studied in many different classes of groups; some of the results will be mentioned in the main part of this survey.

Whereas the subgroup membership problem is a restriction of the submonoid membership problem, one also finds a generalization of the submonoid membership problem in the literature. The class of *rational subsets* of a group G is the smallest class that can be obtained from finite subsets of G using three set operations: union, product (i.e., $S \cdot T = \{gh : g \in S, h \in T\}$ for subsets $S, T \subseteq G$) and Kleene star $\langle S \rangle^*$, where $\langle S \rangle^*$ is the submonoid generated by S (often it is denoted by S^*). Alternatively, one can define rational subsets of G using finite automata whose transitions are labelled with elements of G. Clearly, a rational subset of a finitely generated group G has a finitary representation (by a regular expression or an automaton) which makes the *rational subset membership problem* well-defined. The rational subset membership problem can be traced back to a paper of Benois from 1969 [7], where she proved decidability for free groups. Gilman [23] independently rediscovered Benois' approach in 1984 and extended it to groups with a monadic confluent presentation. Grunschlag showed in 1999 that decidability of the rational subset membership problem is preserved by finite extensions and proved decidability for finitely generated abelian groups based on classical work of Eilenberg and Schützenberger. Kambites, Silva, and Steinberg took up the work of Grunschlag and showed in 2007 that rational subsets membership is decidable for the fundamental group of a graph of groups with finite edge groups and vertex groups with decidable rational subset membership problems [29]. This paper marks the starting point for a deeper investigation of the rational subset membership problem in many different classes of groups; see the main part of this survey for results obtained after 2007. In 2014, Myasnikov, Nikolaev and Ushakov introduced with the *knapsack problem* another special case of the rational subset membership problem [49]. It generalizes the classical knapsack problem over the cyclic group $\mathbb{Z}$ to a non-commutative setting.

Group theoretic setting. Before we introduce the above mentioned problems formally, we first set up the group theoretical context. Consider a group G. For a subset $\Sigma \subseteq G$ we denote with $\langle \Sigma \rangle$ the *subgroup* generated by Σ. It is the smallest subgroup (with respect to inclusion) of G that contains Σ. We will also consider the *submonoid* $\langle \Sigma \rangle^* \subseteq G$ and *subsemigroup* $\langle \Sigma \rangle^+ \subseteq G$ generated by Σ. Note that $\langle \Sigma \rangle^*$ ($\langle \Sigma \rangle^+$) is the set of all finite products $a_1 a_2 \cdots a_n \in G$ with $a_1, \ldots, a_n \in \Sigma$ and $n \geq 0$ ($n \geq 1$). A group G is *finitely generated* (f.g. for short) if there is a finite subset $\Sigma \subseteq G$ such that $G = \langle \Sigma \rangle$. In this situation, we say that Σ is a finite generating set of G. W.l.o.g. one can assume that for every $a \in \Sigma$, the inverse a^{-1} also belongs to Σ. For convenience, we will always assume this. It implies that $\langle \Sigma \rangle = \langle \Sigma \rangle^* = G$, i.e., there is a surjective monoid homomorphism $\pi : \Sigma^* \to G$ with $\pi(a) = a$ for all $a \in \Sigma$ (Σ^* is the free monoid generated by Σ) that we call the *evaluation homomorphism*. We also say that the word $w \in \Sigma^*$ represents the group element $\pi(w)$. For $g \in G$ we also write $|g|$ for the minimal length of a word in $\pi^{-1}(g)$. Here, we assume a fixed $\pi : \Sigma^* \to G$. We only deal with f.g. groups in this survey.

Formal definition of the membership problems. We now define the computational problems that will be considered in this survey. For this we fix a f.g. group G together with an evaluation homomorphism $\pi : \Sigma^* \to G$ with Σ finite. Formally, the algorithmic problems that we introduce below depend on the choice of π. On the other hand, for the decidability and computational complexity of each of the following problems, the concrete choice of the set of generators only plays a minor role. If we take another generating set, the resulting problem is equivalent to the original problem with respect to logspace reductions.

Subgroup membership problem for G, $\mathsf{SGM}(G)$ for short:
- Input: words $w, w_1, \ldots, w_n \in \Sigma^*$
- Question: Does $\pi(w)$ belong to the subgroup $\langle \pi(w_1), \ldots, \pi(w_n) \rangle$?

The subgroup membership problem is also known as the generalized word problem or occurrence problem. It naturally generalizes to submonoids:

Submonoid membership problem for G, $\mathsf{SMM}(G)$ for short:
- Input: words $w, w_1, \ldots, w_n \in \Sigma^*$
- Question: Does $\pi(w)$ belong to the submonoid $\langle \pi(w_1), \ldots, \pi(w_n) \rangle^*$?

One might also consider the *subsemigroup membership problem* by replacing the submonoid $\langle \pi(w_1), \ldots, \pi(w_n) \rangle^*$ by the subsemigroup $\langle \pi(w_1), \ldots, \pi(w_n) \rangle^+$. But there is no real difference between the submonoid and the subsemigroup membership problem: For all $g \in G$ and $S \subseteq G$ we have $g \in \langle S \rangle^+$ iff $gh^{-1} \in \langle S \rangle^*$ for some $h \in S$. Vice versa, $g \in \langle S \rangle^*$ iff $g \in \langle S \rangle^+$ or $g \in \langle 1 \rangle^+$.

The submonoid membership problem can be further generalized to the rational subset membership problem. For a nondeterministic finite automaton $\mathcal{A}$ over an alphabet Σ we write $L(\mathcal{A}) \subseteq \Sigma^*$ for the language accepted by $\mathcal{A}$.

Rational subset membership problem for G, $\mathsf{RatM}(G)$ for short:
- Input: word $w \in \Sigma^*$ and a nondeterministic finite automaton $\mathcal{A}$ over the alphabet Σ
- Question: Does $\pi(w)$ belong to $\pi(L(\mathcal{A}))$?

The subsets $\pi(L(\mathcal{A})) \subseteq G$ with $\mathcal{A}$ a nondeterministic finite automaton are exactly the rational subsets of G defined above. Since for a rational subset $S \subseteq G$ and an element $g \in G$ the set $g^{-1}S$ is rational too, one can restrict to the case $w = \varepsilon$ (i.e., $\pi(w) = 1$) in the rational subset membership problem. The same restriction imposed on the subsemigroup membership problem defines the so-called *identity problem*:

Identity problem for G, $\mathsf{Id}(G)$ for short:
- Input: words $w, w_1, \ldots, w_n \in \Sigma^*$
- Question: Does 1 belong to the subsemigroup $\langle \pi(w_1), \ldots, \pi(w_n) \rangle^+$?

The identity problem was first studied by Choffrut and Karhumäki [14] for matrix groups. Bell and Potapov proved that the identity problem is undecidable for $\mathsf{SL}_4(\mathbb{Z})$ [6], whereas the problem is NP-complete for $\mathsf{GL}_2(\mathbb{Z})$ [5].

Another special case of the rational subset membership problem that has received a lot of attention in recent years is the *knapsack problem*:

Knapsack problem for G, $\mathsf{KS}(G)$ for short:
- Input: words $w, w_1, \ldots, w_n \in \Sigma^*$
- Question: Does $\pi(w)$ belong to $\langle \pi(w_1)\rangle^* \langle \pi(w_2)\rangle^* \cdots \langle \pi(w_n)\rangle^*$?

In other words, $\mathsf{KS}(G)$ is the membership problem for products of cyclic submonoids. A natural variant is the membership problem for products of cyclic subgroups. It can be reduced to $\mathsf{KS}(G)$, since $\langle g\rangle = \langle g\rangle^* \langle g^{-1}\rangle^*$.

The most general problem that we consider is *context-free membership*:

Context-free membership problem for G, $\mathsf{CFM}(G)$ for short:
- Input: word $w \in \Sigma^*$ and a context-free grammar $\mathcal{G}$ over the alphabet Σ
- Question: Does $\pi(w)$ belong to $\pi(L(\mathcal{G}))$?

In Sections 2–10 we will give an overview on decidability and complexity results for the above mentioned problems in different classes of groups. We assume that the reader has some basic knowledge in group theory, computability, complexity theory and formal language theory. Occasionally, we use group presentations to describe groups. For a finite alphabet Γ and set of relators $R \subseteq (\Gamma \cup \Gamma^{-1})^*$ we write $\langle \Gamma \mid R\rangle$ for the corresponding group; it is the quotient of the free group $F(\Gamma)$ by the normal closure of R. For better readability we also write relators as equations: an equation $u = v$ corresponds to the relator uv^{-1}.

There are two related surveys on membership problems in group theory: [17,35]. The paper [35] focuses on rational subset membership. The present survey can be seen as an updated version of [35]. The more recent work [17] of Dong focuses on algorithmic problems for subsemigroups of groups including the subsemigroup membership problem and the identity problem. References on the complexity of word problems can be found in [38].

2 Virtually abelian groups

A group is *virtually abelian* if it has an abelian subgroup of finite index. In these groups, most algorithmic problems are decidable, and this holds also for membership problems:

Theorem 1. $\mathsf{CFM}(G)$ *is decidable for every f.g. virtually abelian group* G.

Proof (sketch). For the case that G is f.g. abelian, one can use Parikh's theorem to reduce $\mathsf{CFM}(G)$ to $\mathsf{RatM}(G)$. Since G is abelian, $\mathsf{RatM}(G)$ is decidable [25]. It therefore suffices to show that if G is a finite-index subgroup of H and $\mathsf{CFM}(G)$ is decidable, then also $\mathsf{CFM}(H)$ is decidable. One can show this with the same arguments used for rational subset membership in [35]. The crucial facts used in [35] are: (i) the word problem (viewed as the set of all words that represent the group identity) of H can be obtained by a rational transduction from the word problem for G and (ii) the class of rational languages is closed under rational transductions. But (ii) also holds for context-free languages [9, Cor. 4.2]. $\square$

3 Free groups and groups containing free monoids

To the knowledge of the author, the earliest result on the rational subset membership problem is due to Benois:

Theorem 2 ([7]). $\mathsf{RatM}(F)$ *is decidable for every f.g. free group F and can be solved in polynomial time.*

The context-free membership problem is undecidable for free groups of rank at least two. In fact, the following more general result holds (note that the free group of rank 2 contains a copy of the free monoid $\{a, b\}^*$):

Theorem 3. *If a f.g. group G contains a copy of a free monoid $\{a, b\}^*$ then* $\mathsf{CFM}(G)$ *is undecidable.*

Proof. Assume that the free monoid $\{a, b\}^*$ is a submonoid of G. In particular, we assume that $a, b \in G$. We reduce Post's correspondence problem (PCP) to $\mathsf{CFM}(G)$. Let $\Pi = \{(u_i, v_i) : 1 \leq i \leq k\}$ be an instance of PCP with $u_i, v_i \in \{a, b\}^*$. Consider the context-free grammar $\mathcal{G}$ with the productions $S \to u_i v_i^{-1}$ and $S \to u_i S v_i^{-1}$ for $1 \leq i \leq k$. We have $1 \in \pi(L(\mathcal{G}))$ iff Π has a solution. □

Rosenblatt proved that a f.g. solvable group is either virtually nilpotent or contains a copy of $\{a, b\}^*$ [56]. Hence, the context-free membership problem for a f.g. solvable group that is not virtually nilpotent is undecidable.

4 Hyperbolic groups

Hyperbolic groups are f.g. groups whose Cayley graph satisfies a certain condition that is motivated from hyperbolic geometry (geodesic triangles are δ-thin for a constant $\delta > 0$). They are one of the most important classes of groups in geometric group theory and have some nice algorithmic properties; for instance the word and conjugacy problem can be solved in linear time. In contrast, the subgroup membership problem is much harder by the following result of Rips:

Theorem 4 ([53]). *There is a hyperbolic group G with* $\mathsf{SGM}(G)$ *undecidable.*

Positive results are known for the knapsack problem in hyperbolic groups. The complexity class LogCFL consists of all languages that are logspace-reducible to a context-free language; it is a subset of $\mathsf{P} \cap \mathsf{DSPACE}(\log^2 n)$.

Theorem 5 ([36,49]). *For every hyperbolic group G,* $\mathsf{KS}(G)$ *is decidable and belongs to the complexity class* LogCFL.

It is shown in [49] for every hyperbolic group G there is a polynomial $p(x)$ such that if $g \in \langle g_1 \rangle^* \langle g_2 \rangle^* \cdots \langle g_n \rangle^*$ for $g, g_1, \ldots, g_n \in G$ then there exist exponents $e_1, \ldots, e_n \in \mathbb{N}$ such that $g = g_1^{e_1} g_2^{e_2} \cdots g_n^{e_n}$ and $e_i \leq p(|g| + |g_1| + \cdots + |g_n|)$ for all i. This allows to reduce the knapsack problem for G to the *acyclic rational subset membership problem* for G, where the input automaton must be acyclic.

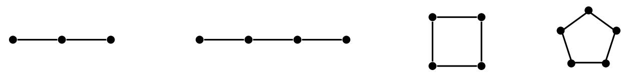

Fig. 1. From left to right: the graphs P3, P4, C4 and C5.

The latter problem is shown to be in LogCFL in [36] using a result from [12] saying that context-sensitive languages where all productions are length increasing (so-called growing-context sensitive languages) can be accepted by nondeterministic one-way Turing machines working in polynomial time and logarithmic space and equipped with an additional pushdown store (the space used on the pushdown store does not count to the logarithmic space bound). These machines define LogCFL even without the one-way restriction. The word problem for a hyperbolic is known to be growing context-sensitive [38]. It is also shown in [36] that the knapsack problem for the free group F_2 of rank two is already LogCFL-complete.

Decidability of the identity problem for hyperbolic groups seems to be open.

5 Graph groups

Let (Γ, I) be a finite graph, where Γ is the set of nodes and $I \subseteq \Gamma \times \Gamma$ is the symmetric and irreflexive edge relation. In the following we just speak of a graph. Figure 1 shows some graphs that will appear below. The *graph group* $\mathsf{G}(\Gamma, I)$ is the finitely presented group $\langle \Gamma \mid ab = ba \text{ for all } (a,b) \in I \rangle$. Graph groups are also known as right-angled Artin groups. They are linear; therefore their word problems can be solved in logspace. For the subgroup membership problem the situation is quite complicated. Note that $F_2 \times F_2$ (the direct product of two free groups of rank 2) is isomorphic to $\mathsf{G}(\mathsf{C4})$. Mihaĭlova proved the following:

Theorem 6 ([48]). $\mathsf{SGM}(\mathsf{G}(\mathsf{C4})) = \mathsf{SGM}(F_2 \times F_2)$ *is undecidable.*

A positive decidability result is known for chordal graphs, i.e., graphs that do not contain a cycle of length at least four as an induced subgraph.

Theorem 7 ([30]). $\mathsf{SGM}(\mathsf{G}(\Gamma, I))$ *is decidable if the graph* (Γ, I) *is chordal.*

In [30], Theorem 7 is stated as a corollary of a more general result saying that the subgroup membership problem is decidable for the fundamental group of a graph of groups where all vertex and edge groups are polycyclic-by-finite. An alternative proof of this result was given in [40].

A characterization of those graph groups having a decidable subgroup membership problem is not known. In particular, it is open, whether $\mathsf{G}(\mathsf{C5})$ has a decidable subgroup membership problem. For the rational subset membership problem and the submonoid membership problem, an exact characterization of the decidable cases is known. Graphs that neither contain P4 nor C4 as an induced subgraph are also known as *transitive forests.*

Theorem 8 ([39]). *Let* (Γ, I) *be a graph. The following are equivalent:*

- *(Γ, I) is a transitive forest.*
- $\mathsf{SMM}(\mathsf{G}(\Gamma, I))$ *is decidable.*
- $\mathsf{RatM}(\mathsf{G}(\Gamma, I))$ *is decidable.*

For the undecidability part of Theorem 8, it suffices to show that $\mathsf{SMM}(\mathsf{G}(\mathsf{P4}))$ and $\mathsf{SMM}(\mathsf{G}(\mathsf{C4}))$ are undecidable. The latter follows from Theorem 6. Undecidability of $\mathsf{SMM}(\mathsf{G}(\mathsf{P4}))$ is shown in [39] by using a result on trace monoids from [1]. Trace monoids can be viewed as the monoid counterparts of graph groups. Given a finite graph (Γ, I) as above, the corresponding trace monoid $\mathsf{M}(\Gamma, I)$ is the quotient of the free monoid Σ^* by the congruence generated by all pairs (ab, ba) with $(a, b) \in I$. Rational subsets of $\mathsf{M}(\Gamma, I)$ are defined as for groups. The *disjointness problem* for rational subsets of $\mathsf{M}(\Gamma, I)$ asks whether $K \cap L \neq \emptyset$ for two given rational subsets $K, L \subseteq \mathsf{M}(\Gamma, I)$. It is shown in [1] that the disjointness problem for rational subsets of $\mathsf{M}(\Gamma, I)$ is decidable if and only if (Γ, I) is a transitive forest. Finally, note that $K \cap L \neq \emptyset$ for $K, L \subseteq \mathsf{M}(\Gamma, I)$ if and only if $1 \in KL^{-1}$, where KL^{-1} is viewed as a rational subset of the graph group $\mathsf{G}(\Gamma, I)$. This shows in particular that $\mathsf{RatPM}(\mathsf{G}(\mathsf{P4}))$ is undecidable. Finally, $\mathsf{RatPM}(\mathsf{G}(\mathsf{P4}))$ is reduced to $\mathsf{SMM}(\mathsf{G}(\mathsf{P4}))$ in [39].

The decidability part in Theorem 8 was further refined by Haase and Zetzsche who determined the complexity of $\mathsf{RatM}(\mathsf{G}(\Gamma, I))$ for transitive forests:

Theorem 9 ([27]). *Let (Γ, I) be a transitive forest. Then* $\mathsf{RatM}(\mathsf{G}(\Gamma, I))$ *is*
- NL*-complete if (Γ, I) is a clique,*
- P*-complete if (Γ, I) is a disjoint union of at least two cliques, and*
- NP*-complete if (Γ, I) contains an induced* $\mathsf{P3}$*, i.e., (Γ, I) is not transitive.*

The uniform version of the rational subset membership problem for graph groups, where the transitive forest (Γ, I) is part of the input is $\mathsf{NEXPTIME}$*-complete.*

For the $\mathsf{NEXPTIME}$ upper bound in the uniform case, where the transitive forest (Γ, I) is part of the input, Haase and Zetzsche show that the rational subset membership problem is equivalent to the satisfiability problem for existential Presburger arithmetic extended by the star operator. The star operator applied to a set $S \subseteq \mathbb{N}^k$ yields the submonoid of the additive monoid $(\mathbb{N}^k, +)$ generated by S. It is shown in [27] that the satisfiability problem for existential Presburger arithmetic with the star operator is $\mathsf{NEXPTIME}$-complete. In addition, if the nesting depth of the star operator in the input formula is bounded by a fixed constant then the satisfiability problem becomes NP-complete (and hence has the same complexity as the satisfiability problem for existential Presburger arithmetic without the star operator). This yields the NP upper bound in Theorem 9. For the NP lower bound, the subset sum problem is reduced to $\mathsf{RatM}(F_2 \times \mathbb{Z})$.

A characterization similar to Theorem 9 is also known for the knapsack problem. The complexity class TC^0 is a very small class within LogCFL. It is captured by the problem of counting the number of 1's in a bit string.

Theorem 10 ([45]). $\mathsf{KS}(\mathsf{G}(\Gamma, I))$ *is decidable for every graph (Γ, I) and is*
- TC^0*-complete if (Γ, I) is a clique,*
- LogCFL*-complete if (Γ, I) is a transitive forest but not a clique, and*

– NP-*complete if* (Γ, I) *is not a transitive forest.*

For the NP upper bound the line of arguments in [45] goes as follows: Consider a graph group $G = \mathsf{G}(\Gamma, I)$. It is shown in [45] that if $g \in \langle g_1\rangle^*\langle g_2\rangle^* \cdots \langle g_n\rangle^*$ for $g, g_1, \ldots, g_n \in G$ then there exist $e_1, \ldots, e_n \in \mathbb{N}$ such that $g = g_1^{e_1} g_2^{e_2} \cdots g_n^{e_n}$ and every e_i is exponentially bounded in $|g| + |g_1| + \cdots + |g_n|$. One can therefore nondeterministically guess in polynomial time the binary encodings of exponents $e_1, \ldots, e_n$. It then remains to verify whether $g = g_1^{e_1} g_2^{e_2} \cdots g_n^{e_n}$ holds. This is an instance of the so-called compressed word problem for the graph group G, which can be solved in polynomial time [34]. The LogCFL upper bound in Theorem 10 for the case that (Γ, I) is a transitive forest, follows the same line of arguments that was sketched for hyperbolic groups in Section 4.

Let us also mention that the uniform knapsack problem for graph groups, where the graph is part of the input, is NP-complete by a recent result form [44].

6 Nilpotent groups and polycyclic groups

Polycyclic groups are solvable groups where every subgroup is finitely generated. Mal'cev proved in [46] the following result (see [2] for an alternative proof):

Theorem 11 ([46]). $\mathsf{SGM}(G)$ *is decidable for every polycyclic group* G.

To the knowledge of the author, the complexity of the subgroup membership problem for polycyclic groups is open. For the smaller class of f.g. nilpotent groups, the complexity is very low:

Theorem 12 ([50]). $\mathsf{SGM}(G)$ *belongs to* TC^0 *for every f.g. nilpotent group* G.

Recently, the identity problem for nilpotent groups attracted a lot of attention. Dong [19] showed that the identity problem is decidable for all nilpotent groups of class at most 10. Shafrir then solved the general case:

Theorem 13 ([57]). $\mathsf{Id}(G)$ *belongs to* P *for every f.g. nilpotent group* G.

The main ingredient in [57] is the following result, where $G' = [G, G]$ is the commutator subgroup of G: If G is a f.g. nilpotent group and M is a submonoid of G such that MG' is a finite index subgroup of G then M is also a finite index subgroup of G. Moreover, if $MG' = G$ then $M = G$. For the case that M is a subgroup G this was known before. Shafrir combines this result with linear programming techniques (Farkas' lemma) to prove Theorem 13. With similar techniques, one can show Theorem 13 also using [10, Proposition 2.5], whose proof will appear in a forthcoming paper of Bodart, Ciobanu, and Metcalfe.

Somehow surprisingly, the knapsack problem and the submonoid membership problem are in general undecidable already for nilpotent groups of class 2. The following two results can be shown using reductions from Hilbert's 10th problem.

Theorem 14 ([54]). *There is a f.g. class-2 nilpotent group* G *with* $\mathsf{SMM}(G)$ *undecidable.*

Theorem 15 ([32]). *There is a f.g. class-2 nilpotent group G with* $\mathsf{KS}(G)$ *undecidable.*

For nilpotent groups of small *Hirsch length* further decidability were recently shown by Shafrir. The Hirsch length $h(G)$ of a nilpotent group G is the sum of the ranks of all the successive abelian factor groups in the lower central series.

Theorem 16 ([58]). $\mathsf{KS}(G)$ *is decidable for every f.g. nilpotent group G with $h([G,G]) = 1$, which means that $[G,G] \cong \mathbb{Z} \times A$ for a finite abelian group A.*

The proof of Theorem 16 is based on a short reduction of $\mathsf{KS}(G)$ to the problem of whether a system consisting of (i) a single quadratic Diophantine equation plus (ii) an arbitrary number of linear equations has an integer solution. Decidability of this special case of Hilbert's 10th problem is shown in [20,24].

Theorem 17 ([58]). $\mathsf{SMM}(G)$ *is decidable for every f.g. nilpotent group G with $h([G,G]) \leq 2$.*

Shafrir uses a result of Bodart [10] that allows to reduce membership in a submonoid $M \subseteq G$ (for G a f.g. nilpotent group) to membership in products $\langle S_1\rangle^*\langle S_2\rangle^* \cdots \langle S_n\rangle^*$, where all S_i are finite and are contained in a subgroup $H \leq G$ with $h([H,H]) < h([G,G])$. Here, H depends on M. Then Shafrir shows, using a combinatorial lemma on integer sequences with bounded gaps, that in a f.g. nilpotent group H with $h([H,H]) = 1$, every f.g. submonoid is effectively a product of cyclic submonoids. Hence, the product $\langle S_1\rangle^*\langle S_2\rangle^* \cdots \langle S_n\rangle^*$ can be replaced by a product of cyclic submonoids, which allows to apply Theorem 16.

Theorems 16 and 17 generalize previous results for Heisenberg groups $H_n(\mathbb{Z})$ from [32] (for knapsack) and [15] (for submonoid membership). The Heisenberg group $H_n(\mathbb{Z})$ consists of all n-dimensional upper triangular integer matrices such that all diagonal entries are 1 and all non-diagonal entries that are not in the top-most row or the right-most column are 0. Note that $[H_n(\mathbb{Z}), H_n(\mathbb{Z})] \cong \mathbb{Z}$; therefore Theorems 16 and 17 apply. For $n = 3$, even rational subset membership is decidable by the following result of Bodart.

Theorem 18 ([10]). $\mathsf{RatM}(H_3(\mathbb{Z}))$ *is decidable.*

Bodart proved this result by reducing $\mathsf{RatM}(H_3(\mathbb{Z}))$ to $\mathsf{KS}(H_3(\mathbb{Z}))$. For this he shows that every rational subset of $H_3(\mathbb{Z})$ is effectively the image of a *bounded* regular language L. A language is called bounded if it is contained in a set $w_1^* w_2^* \cdots w_k^*$ for words $w_1, \dots, w_k$. One finally obtains a finite number of knapsack instances from a classical result of formal language theory saying that a regular language is bounded iff it is a finite union of languages $v_0 w_1^* v_1 w_2^* \cdots v_{k-1} w_k^* v_k$.

It is open, whether Theorem 18 generalizes to all Heisenberg groups $H_n(\mathbb{Z})$. It is also open whether $\mathsf{CFM}(H_3(\mathbb{Z}))$ is decidable. We have already observed that every f.g. solvable group that is not virtually nilpotent has an undecidable context-free membership problem (Section 3). It is known that every f.g. virtually nilpotent group that is not virtually abelian contains a copy of $H_3(\mathbb{Z})$ [28, proof of Theorem 12]. We obtain the following conditional characterization: assuming that $\mathsf{CFM}(H_3(\mathbb{Z}))$ is undecidable, a f.g. solvable group G is virtually abelian if and only if $\mathsf{CFM}(G)$ is decidable.

7 Metabelian groups

A group G is *metabelian* if its commutator subgroup $[G, G]$ is abelian.

Theorem 19 ([3,55]). $\mathsf{SGM}(G)$ *is decidable for every f.g. metabelian group* G.

Both papers [3,55] actually show a more general result: if G is f.g. abelian-by-nilpotent then $\mathsf{SGM}(G)$ is decidable. The complexity of the subgroup membership problem for metabelian groups seems to be open. For the identity problem, Dong showed the following result. The proof is very involved and combines techniques from convex geometry, graph theory, algebraic geometry, and number theory:

Theorem 20 ([18]). $\mathsf{Id}(G)$ *is decidable for every f.g. metabelian group* G.

Theorems 19 and 20 are all general positive results for membership problems in f.g. metabelian groups that the author is aware of. On the side of undecidability, the following is known.

Theorem 21 ([42,43]). *There are f.g. metabelian groups* G *with* $\mathsf{SMM}(G)$ *undecidable. Examples are the wreath product* $\mathbb{Z} \wr \mathbb{Z}$ *and the free metabelian group* M_2 *generated by two elements.*

Undecidability of $\mathsf{SMM}(M_2)$ is shown in [43] by a reduction from a tiling problem for the Euclidean plane, whereas undecidability of $\mathsf{SMM}(\mathbb{Z} \wr \mathbb{Z})$ is shown in [43] by a reduction from the halting problem for 2-counter machines.

Also the knapsack problem is in general undecidable for f.g. metabelian groups by Theorem 15, because nilpotent groups of class two are metabelian. On the other hand, free metabelian groups have a decidable knapsack problem; this is a special case of Theorem 26 below. Also the knapsack problem for $\mathbb{Z} \wr \mathbb{Z}$ is decidable. This follows from a more general result: $\mathsf{KS}(G)$ is decidable for every co-context-free group G [32]. A f.g. group G with the evaluation homomorphism $\pi : \Sigma^* \to G$ is *co-context-free* if the co-word problem $\{w \in \Sigma^* : \pi(w) \neq 1\}$ is context-free [28]. The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is co-context-free [28, Theorem 10]. Another interesting example of a co-context-free group is Thompson's group F.

8 Wreath products

We mentioned above that $\mathsf{SMM}(\mathbb{Z} \wr \mathbb{Z})$ is undecidable [43]. In the same paper, the following positive result was shown:

Theorem 22 ([43]). *If* G *is a finite group and* H *is a f.g. virtually free group then* $\mathsf{RatM}(G \wr H)$ *is decidable.*

The proof of this result in [43] uses techniques from the theory of well-quasi orders. Due to this, the algorithm in [43] is not primitive recursive.

It seems to be difficult to extend Theorem 22 beyond the case where H is virtually free. For $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ the following undecidability result is shown in [42] using a tiling problem.

Theorem 23 ([42]). *If $G \neq 1$ is a f.g. group then* $\mathsf{RatM}(G \wr \mathbb{Z}^2)$ *is undecidable.*

In contrast to Theorem 23, Shafrir showed the following result:

Theorem 24 ([52]). *If G is finite and abelian then* $\mathsf{SMM}(G \wr \mathbb{Z}^2)$ *is decidable.*

Shafrir's proof can be found in the Bachelor thesis [52], which is based on an unpublished draft of Shafrir. He shows that for any wreath product $W = G \wr \mathbb{Z}^2$ with G finite, $\mathsf{SMM}(W)$ can be reduced to $\mathsf{SGM}(W)$. If, in addition, G is abelian then W is metabelian and $\mathsf{SGM}(W)$ is decidable by Theorem 19.

Together, Theorems 23 and 24 yield examples of groups G where $\mathsf{RatM}(G)$ is undecidable but $\mathsf{SMM}(G)$ is decidable. The existence of such groups solved an open problem from [41], where it is shown that if G is a f.g. group with more than one end then there is a computable reduction from $\mathsf{RatM}(G)$ to $\mathsf{SMM}(G)$. Bodart [10] recently gave another example of a group G, where $\mathsf{RatM}(G)$ is undecidable but $\mathsf{SMM}(G)$ is decidable. His example is a f.g. nilpotent group G of class 2.

A corollary of these results is that decidability of the submonoid membership problem is not preserved by free products (in contrast to subgroup membership and rational subset membership): If it would be, then $\mathsf{SMM}(G * \mathbb{Z})$ would be decidable, where G is such that $\mathsf{RatM}(G)$ is undecidable and $\mathsf{SMM}(G)$ is decidable. Since $G * \mathbb{Z}$ has infinitely many ends, $\mathsf{RatM}(G * \mathbb{Z})$ and hence $\mathsf{RatM}(G)$ would be decidable by [41], which is a contradiction.

A characterization of the class of wreath products $G \wr H$ with a decidable knapsack problem can be found in [8]. The characterization is a bit technical and uses knapsack variants for G and H; we refer the reader to [8].

9 Solvable groups

Metabelian groups are exactly the solvable groups of derived length 2. There is no hope to get decidability results for arbitrary solvable groups of derived length 3: Kharlampovich constructed a finitely presented solvable group of derived length 3 with an undecidable word problem [31]. For the subgroup membership problem, undecidability is already encountered for free solvable groups. Let $S_{c,r}$ be the free solvable group of derived length c generated by r elements.

Theorem 25 ([59]). $\mathsf{SGM}(S_{3,2})$ *is undecidable.*

The knapsack problem for free solvable groups turns out to be easier:

Theorem 26 ([21]). *For every $c, r \geq 2$,* $\mathsf{KS}(S_{c,r})$ *is decidable and* NP*-complete.*

The proof of Theorem 26 exploits the Magnus embedding theorem that allows to embed a free solvable group into an iterated wreath product $\mathbb{Z}^m \wr (\mathbb{Z}^m \wr (\mathbb{Z}^m \cdots))$. For such a wreath product one can show that if $g \in \langle g_1 \rangle^* \langle g_2 \rangle^* \cdots \langle g_n \rangle^*$ then $g = g_1^{e_1} g_2^{e_2} \cdots g_n^{e_n}$, where every $e_i \in \mathbb{N}$ is exponentially bounded in $|g| + |g_1| + \cdots + |g_n|$.

Solvable matrix groups are another important class of solvable groups with a more amenable algorithmic theory. Note that by Tits alternative a f.g. matrix group is either virtually solvable or contains a copy of the free group of rank 2.

Theorem 27 ([33]). $\mathsf{SGM}(G)$ *is decidable for every f.g. virtually solvable matrix group over the field of algebraic numbers.*

Theorem 28 ([11]). $\mathsf{Id}(G)$ *is decidable for every f.g. virtually solvable matrix group over the field of algebraic numbers.*

Theorem 28 generalizes the decidability statement in Theorem 13. The proof of Theorem 28 extends some of the key results from [10,18,57].

10 Baumslag-Solitar groups

For $p, q \geq 1$, the *Baumslag-Solitar group* $\mathsf{BS}_{p,q} = \langle a, t \mid t^{-1}a^p t = a^q \rangle$ is a one-relator group that was introduced by Baumslag and Solitar in [4], where they showed that $\mathsf{BS}_{2,3}$ is non-Hopfian (i.e., $\mathsf{BS}_{2,3}$ is isomorphic to a proper quotient of $\mathsf{BS}_{2,3}$). The word problem for every Baumslag-Solitar group $\mathsf{BS}_{p,q}$ can be solved in logarithmic space. The following result is a corollary of a more general result from [30] that has already mentioned after Theorem 7.

Theorem 29 ([30]). $\mathsf{SGM}(\mathsf{BS}_{p,q})$ *is decidable for all for* $p, q \geq 1$.

The groups $\mathsf{BS}_{1,q}$ is also metabelian, so $\mathsf{Id}(\mathsf{BS}_{1,q})$ is decidable by Theorem 20. In addition, the following two positive results hold:

Theorem 30 ([13]). $\mathsf{RatM}(\mathsf{BS}_{1,q})$ *for* $q \geq 2$ *is* PSPACE*-complete.*

Theorem 31 ([22]). $\mathsf{KS}(\mathsf{BS}_{1,q})$ *for* $q \geq 2$ *is* NP*-complete.*

In their proof of Theorem 30 the authors use a particular word representation of elements of $\mathsf{BS}_{1,q}$, which they call the pointed expansion. It is well known that $\mathsf{BS}_{1,q}$ is isomorphic to the semidirect product $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$, where $\mathbb{Z}[1/q]$ is the additive group of all rationals of the form zq^i where $z, i \in \mathbb{Z}$ and $\mathbb{Z}$ acts on $\mathbb{Z}[1/q]$ by $j \cdot zq^i = zq^{i+j}$ for $i, j, z \in \mathbb{Z}$. Hence, one can represent an element $g \in \mathsf{BS}_{1,q} \cong \mathbb{Z}[1/q] \rtimes \mathbb{Z}$ by a pair $(\pm a_k a_{k+1} \cdots a_{0\bullet} a_{-1} a_{-2} \cdots a_{-\ell}, i)$, where $\pm a_k a_{k+1} \cdots a_{0\bullet} a_{-1} a_2 \cdots a_\ell$ is the q-ary representation of the $\mathbb{Z}[1/q]$-part (hence, $a_k, \ldots, a_{-\ell} \in [0, q-1]$) and $i \in [-\ell, k]$ is the $\mathbb{Z}$-part. Uniqueness of this representation can be obtained by choosing the interval $[-\ell, k]$ minimal. The pointed expansion of G is then obtained by taking the word $\pm a_k a_{k+1} \cdots a_{0\bullet} a_{-1} a_{-2} \cdots a_{-\ell}$ and marking a_i with a special marker. The main result of [22] states that for a rational subset $S \subseteq G$ the set $\mathsf{pe}(S)$ of all pointed expansions of elements from S is effectively a regular language. Moreover, from a given automaton for S one can construct in polynomial space a certain succinct description of an automaton for $\mathsf{pe}(S)$. This suffices in order to get the PSPACE upper bound in Theorem 30. The PSPACE lower bound is shown by a reduction from the intersection nonemptiness problem for finite automata.

For the proof of Theorem 31, $\mathsf{KS}(\mathsf{BS}_{1,q})$ is reduced in [22] to the existential fragment of Büchi arithmetic. Büchi arithmetic is the first-order theory of the structure $(\mathbb{Z}, +, \geq, 0, V_q)$, where V_q is the function that maps the the integer n

to the largest power of q that divides n. Here, $q \geq 2$ is a fixed integer. The existential fragment of Büchi arithmetic was shown to be in NP in [26].

Let us finally remark that since $\mathsf{BS}_{1,q}$ contains a copy of the free submonoid $\{a, b\}^*$ ($\mathsf{BS}_{1,q}$ is solvable but not virtually nilpotent), the context-free subset membership problem for $\mathsf{BS}_{1,q}$ is undecidable.

11 Open problems

We conclude with a list of open problems. Some of them were already mentioned in the main part of the paper.

1. Is $\mathsf{CFM}(H_3(\mathbb{Z}))$ decidable?
2. Is there any non-virtually-abelian group G such that $\mathsf{CFM}(G)$ is decidable?
3. Is $\mathsf{RatM}(H_n(\mathbb{Z}))$ decidable for all n?
4. Is $\mathsf{Id}(G)$ decidable for every hyperbolic group G?
5. For which graph groups G is $\mathsf{SGM}(G)$ decidable?
6. Is $\mathsf{RatM}(\mathsf{BS}_{p,q})$ decidable for all p, q? What about $\mathsf{SMM}(\mathsf{BS}_{p,q})$?
7. Is $\mathsf{SGM}(\mathsf{SL}_3(\mathbb{Z}))$ decidable? What about $\mathsf{Id}(\mathsf{SL}_3(\mathbb{Z}))$ and $\mathsf{KS}(\mathsf{SL}_3(\mathbb{Z}))$?
8. Is there a group G such that $\mathsf{SMM}(G)$ is decidable and $\mathsf{KS}(G)$ is undecidable?
9. Is SMM (RatM) decidable for fundamental groups of closed orientable two-dimensional manifolds of genus at least two?

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