Priority and Maximal Progress are completely axiomatisable (extended abstract)

Holger Hermanns[1] and Markus Lohrey[2]

[1] Informatik VII, Universität Erlangen, Germany
e-mail: hrherman@@informatik.uni-erlangen.de
[2] Institut für Informatik, Universität Stuttgart, Germany
e-mail: lohreyms@@informatik.uni-stuttgart.de

Abstract. During the last decade, CCS has been extended in different directions, among them priority and real time. One of the most satisfactory results for CCS is Milner's complete proof system for observational congruence [?]. Observational congruence is fair in the sense that it is possible to escape divergence, reflected by an axiom $\mathrm{rec}X.(\tau.X + P) = \mathrm{rec}X.\tau.P$. In this paper we discuss observational congruence in the context of interactive Markov chains, a simple stochastic timed variant CCS with maximal progress. This property implies that observational congruence becomes unfair, i.e. it is not always possible to escape divergence. This problem also arises in calculi with priority. So, completeness results for such calculi modulo observational congruence have been unknown until now. We obtain a complete proof system by replacing the above axiom by a set of axioms allowing to escape divergence by means of a silent alternative. This treatment can be profitably adapted to other calculi.

1 Introduction

One of the outstanding results for CCS [?] is Milner's complete proof system for regular CCS expressions modulo observational congruence [?]. The task of proving completeness is divided into three parts. First, only guarded recursive expressions are considered where guards are visible actions. This means that divergent expressions (that perform an infinite number of silent steps) are excluded. The core of that part is to show that two congruent expressions satisfy the same set of recursive equations. The second important property is that every set of recursive defining equations has a unique solution. Divergent expressions cannot be handled in this way, since, for instance, the recursive equation $X = \tau.X$ has infinitely many solutions. Therefore completeness is obtained by adding further axioms. In particular, a divergent expression can be equated to a non-divergent expressions by applying essentially the axiom

$$\mathrm{rec}X.(\tau.X + P) = \mathrm{rec}X.\tau.P.$$

Walker [?] studies divergence in the context of CCS and observational congruence. The possibility of escaping from divergence is known as fairness. Koomen [?] was the first to define a fair abstraction rule similar to the one above, which will therfore be referred to as the KFAR axiom throughout this paper. Fairness is mostly regarded as a desirable feature. Therefore, the issue of obtaining fairness has been extensively studied in the literature. Baeten *et al.* [**?**] discusses fairness in the context of failure semantics. In [**?**], Bergstra *et al.* introduce a weaker version of fair abstraction (WFAR) that allows to escape divergence only if a silent alternative exists. Fair testing equivalences have been developed in [**?**,**?**].

In recent years, CCS has been extended in different directions, among them priority and real time. Different prioritised process algebras have been developed, among them [**?**,**?**,**?**]. Investigations of observational congruence in the presence of priority have been restricted to finite, i.e. recursion free processes [**?**]. In that approach priority is nicely reflected by the following axiom where $\underline{a}$ has a lower priority than τ:

$$\tau.P + \underline{a}.Q = \tau.P.$$

A variety of timed process algebras have also been proposed, for instance [**?**,**?**,**?**,**?**,**?**,**?**]. A thorough overview of their basic ingredients is given in [**?**]. Complete proof systems for regular expressions have been obtained for some of these calculi [**?**,**?**,**?**]. One of the typical features of CCS based timed process algebras is a notion of maximal progress, also called minimal delay or τ-urgency. This property says that a system cannot wait if it has something internal to do. It is characterised by the following axiom where $delay(T)$ usually stands for a fixed time delay of length T:

$$\tau.P + delay(T).Q = \tau.P.$$

The concepts of priority and maximal progress arose at different corners in concurrency theory. Weak bisimulation semantics incorporating one of these ingredients, however, have a common feature: Divergence implies unfairness. In particular, the above KFAR axiom is not sound[1]. Thus, KFAR cannot be used to equate divergent expressions to non-divergent expressions. So, completeness is not attainable in this way. But the equation $X = \tau.X$ still has infinitely many solutions. As a consequence, to the best of our knowledge, no complete proof system for observational congruence for regular CCS including either priority or maximal progress has been given until now.

In recent years also stochastic timed calculi have emerged, where delays are not fixed but given by continuous probability distribution functions. This fits neatly to interleaving semantics, if only exponential distributions are considered. Then $delay(T)$ stands for a delay, say t, with mean duration T and distribution $Prob(delay \leq t) = 1 - e^{-\lambda t}$, where the parameter λ is the reciprocal value of T. We mention TIPP of Götz *et al.* [**?**], Hillston's PEPA [**?**], and Bernardo&Gorrieri's EMPA [**?**] as representatives of this approach. Their unifying feature is that their semantics can be transformed into a continuous time Markov chain, a stochastic model widely used for performance evaluation purposes, see e.g. [**?**].

[1] In the timed case, a counterexample is $\mathrm{rec}X.(\tau.X + delay(T).Q)$. KFAR equates this expression to $\mathrm{rec}X.\tau.delay(T).Q$ while maximal progress leads to $\mathrm{rec}X.\tau.X$. Since the latter (using KFAR) can be equated to termination, both expressions obviously describe distinct behaviours.

The contribution of this paper is threefold. *(1)* Concerning ordinary CCS we present a slight modification of Milner's observational congruence that permits to escape divergence *only* if a silent alternative exists. This is exactly the effect of WFAR in the style of [**?**]. This notion of observational congruence is truly contained in Milner's observational congruence. This compares favourably to the treatment of divergence in [**?**] that is incomparable with the original definition. We provide a sound and complete proof system for observational congruence with WFAR on CCS. It is achieved by replacing KFAR by a set of axioms allowing to escape divergence by means of a silent alternative.

(2) We develop that system in order to provide a sound and complete proof system for observational congruence in the calculus of *interactive Markov chains* [**?**], a stochastic timed extension of CCS with maximal progress. This calculus contains ordinary CCS as well as (homogeneous) continuous time Markov chains as proper subalgebras.

(3) Since our treatment of divergence is orthogonal to the stochastic timing aspects we highlight how our proof system can be adapted to a variety of other calculi with either maximal progress or priority for which similar completeness results have been unknown until now.

The stochastic timed calculi of [**?**,**?**,**?**] all attach exponentially distributed delays to actions. Their subtle differences are mainly based on different interpretations of the delay of synchronised actions. With interactive Markov chains, we deviate from these calculi and split delays and actions into two orthogonal parts. This separation is similar to that in timed process algebras as proposed in [**?**,**?**,**?**]. It rules out any ambiguity in the timing of synchronisation.

An extension of interactive Markov chains has been developed to study performance properties of parallel and distributed systems. In [**?**], for instance, it is applied to specify a CSMA/CD protocol stack. The whole system turns out to have 37136 reachable states. It can be proven to be observational congruent to a system with 411 states which can be directly transformed into a Markov chain to study temporal properties of the protocol stack. That case study has indeed initiated our study of equational properties of observational congruence. With the results presented in this paper we have a complete proof system for establishing observational congruence of such systems on the language level.

The paper is organised as follows. Section 2 briefly describes the calculus of interactive Markov chains and defines congruence relations on it. Section 3 presents a set of equational laws that turn out to be sound and complete for observational congruence. Section 4 discusses the relation of our laws to WFAR, ordinary CCS and extensions thereof. Section 5 contains some concluding remarks. The proof of completeness (and of congruence) is quite involved, but has to be omitted due to space constraints. It can be found in [**?**].

2 The Calculus of Interactive Markov Chains

In this section we introduce the basic definitions and properties of the calculus we investigate. It includes a distinct type of prefixing to specify exponentially distributed delays. Instead of a broad introduction into their theory we briefly summarise some important properties enjoyed by exponential distributions. Details can be found in various textbooks, e.g. [?].

(A) An exponential distribution $Prob\{delay \leq t\} = 1 - e^{-\lambda t}$ is characterised by a single parameter λ, a positive real value, usually referred to as the *rate* of the distribution.
(B) Exponential distributions possess the so called *Markov property*. The remaining delay after some time t_0 has elapsed is a random variable with the same distribution as the whole delay: $Prob\{delay \leq t + t_0 \mid delay > t_0\} = Prob\{delay \leq t\}$.
(C) The class of exponential distributions is closed under minimum, which is exponentially distributed with the sum of the rates. More precisely, $Prob\{\min(delay_1, delay_2) \leq t\} = 1 - e^{-(\lambda_1+\lambda_2)t}$ if $delay_1$ ($delay_2$, respectively) is exponentially distributed with rate λ_1 (λ_2).

While property (A) allows a compact syntactic representation of delays in our calculus, the Markov property (B) is important to employ an interleaving semantics. It ensures that distributions of delays do not have to be recalculated after some (causally independent) delay has elapsed. Therefore, the usual expansion law can be applied straightforwardly. This substantially simplifies the definition of parallel composition. Property (C) is decisive for our interpretation of the choice operator in the presence of delays: If all alternatives of a choice involve an exponentially distributed delay the decision is taken as soon as the first of these delays elapses. This finishing delay determines the subsequent behaviour. The time instant of this decision is obviously given by the minimum of distributions. As a consequence of property (C), the overall delay until the decision is taken is exponentially distributed.

After these preliminaries we introduce the calculus of interactive Markov chains. We assume a set of process variables *Var*, a set of actions *Act* containing a distinguished silent action τ and let $I\!R$ denote the set of positive reals. We use λ, μ, ... to range over $I\!R$ and a, b, ... for elements of Act. The basic calculus does not contain parallel composition, we defer the discussion of this operator to Section 4.

Definition 1. *Let $\lambda \in I\!R$, $a \in Act$ and $X \in Var$. We define the language* IMC *as the set of expressions given by the following grammar.*

$$\mathcal{E} ::= 0 \quad | \quad (\lambda).\mathcal{E} \quad | \quad a.\mathcal{E} \quad | \quad \mathcal{E} + \mathcal{E} \quad | \quad X \quad | \quad \mathrm{rec}X.\mathcal{E}$$

The expression $(\lambda).P$ describes a behaviour that will delay its subsequent behaviour P for an exponentially distributed time with a mean duration of $1/\lambda$. The meaning of the other operators is as usual. We use E, F, ... to range over expressions of IMC. With the usual notion of free variables and free and closed expressions we let IMP denote the set of closed expressions, ranged over by P, Q, ..., called processes. $Var(E)$ denotes the set of free variables of E.

A variable X is strongly guarded in an expression E if every occurrence of X in E is strongly guarded, i.e. guarded by a prefix "a." (with $a \neq \tau$) or "(λ).".

Weak guardedness is the same, but includes the prefix "τ.". An expression E is said to be strongly (weakly) guarded, if, for every subexpression of the form $\mathrm{rec}X.E'$, the variable X is strongly (weakly) guarded in E'.

Definition 2. *The set of* well-defined *expressions* $\mathsf{IMC}_{\downarrow}$ *is the smallest subset of* IMC *such that*

- $\mathit{Var} \subseteq \mathsf{IMC}_{\downarrow}$ *and* $0 \in \mathsf{IMC}_{\downarrow}$,
- *if* $E \in \mathsf{IMC}$ *then* $a.E \in \mathsf{IMC}_{\downarrow}$ *and* $(\lambda).E \in \mathsf{IMC}_{\downarrow}$,
- *if* $E \in \mathsf{IMC}_{\downarrow}$ *and* $F \in \mathsf{IMC}_{\downarrow}$ *then* $E + F \in \mathsf{IMC}_{\downarrow}$,
- *if* $E\{\mathrm{rec}X.E/X\} \in \mathsf{IMC}_{\downarrow}$ *then* $\mathrm{rec}X.E \in \mathsf{IMC}_{\downarrow}$.

The complementary subset of IMC *containing all* ill-defined *expressions, will be denoted* $\mathsf{IMC}_{\uparrow}$. *We write* $E\downarrow$ $(E\uparrow)$ *if* $E \in \mathsf{IMC}_{\downarrow}$ $(E \in \mathsf{IMC}_{\uparrow})$.

The semantics of each expression is defined as an equivalence class of transition systems. We define a transition system for each expression below by means of structural operational rules. We define two transition relations, one for actions and one to represent the impact of time. We have taken the liberty to shift the complexity of our calculus from the definition of the transition system towards the definition of equivalences. As a consequence, the operational rules are very simple, whereas the definition of a suitable equivalence becomes more challenging.

Definition 3. *The* action transition *relation* $\longrightarrow\ \subset \mathsf{IMC} \times \mathit{Act} \times \mathsf{IMC}$ *and the* timed transition *relation* $\dashrightarrow\ \subset \mathsf{IMC} \times \mathbb{R} \times \{l, r\}^* \times \mathsf{IMC}$ *are the least relations given by the rules in Figure* **??**.

$$(a^I)\ \frac{}{a.E \xrightarrow{a} E} \qquad (\lambda^M)\ \frac{}{(\lambda).E \overset{\lambda,\,\varepsilon}{\dashrightarrow} E}$$

$$(+^I_l)\ \frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} \qquad (+^M_l)\ \frac{E \overset{\lambda,\,w}{\dashrightarrow} E'}{E + F \overset{\lambda,\,l\,w}{\dashrightarrow} E'}$$

$$(+^I_r)\ \frac{F \xrightarrow{a} F'}{E + F \xrightarrow{a} F'} \qquad (+^M_r)\ \frac{F \overset{\lambda,\,w}{\dashrightarrow} F'}{E + F \overset{\lambda,\,r\,w}{\dashrightarrow} F'}$$

$$(\mathrm{rec}^I)\ \frac{E\{\mathrm{rec}X.E/X\} \xrightarrow{a} E'}{\mathrm{rec}X.E \xrightarrow{a} E'} \qquad (\mathrm{rec}^M)\ \frac{E\{\mathrm{rec}X.E/X\} \overset{\lambda,\,w}{\dashrightarrow} E'}{\mathrm{rec}X.E \overset{\lambda,\,w}{\dashrightarrow} E'}$$

Fig. 1. Operational semantic rules for IMC.

In the rules for timed transitions we use words over $\{l, r\}$ to generate multiple transitions for expressions like $(\lambda).0 + (\lambda).0$ by encoding their different proof trees (ε denotes the empty word). This is known from probabilistic calculi like PCCS [**?**]. The need to represent multiplicities stems from our interpretation of choice in the presence of delays. It is assumed that the decision is taken as soon as the first of the delays elapses. Property (C) implies that this delay is again governed by an exponential distribution given by the sum of the rates. In other words, the behaviour of $(\lambda).0 + (\lambda).0$ is the same as that of $(2\lambda).0$. Thus idempotence of choice does not hold. Our notion of bisimilarity is therefore similar to probabilistic bisimilarity as introduced by Larsen&Skou [**?**] regarding timed transitions. The definition requires to calculate the sum of all rates leading from a single expression into a set of expressions (where the latter set will be an equivalence class of expressions).

Definition 4. *For $E \in \mathsf{IMC}$, $C \subseteq \mathsf{IMC}$ we define the cumulative rate function $\gamma : \mathsf{IMC} \times \wp(\mathsf{IMC}) \longrightarrow I\!R$ as*

$$\gamma(E, C) = \sum_{w \in \{l,r\}^*} \{\lambda \mid \exists F \in C \,:\, E \overset{\lambda,\, w}{\dashrightarrow} F\}.$$

The interrelation of timed and action transitions resulting, for instance, from $(\lambda).P + \tau.Q$ is not evident from the operational rules. From a stochastic perspective, the silent action may happen *instantaneously* because nothing may prevent or delay it. On the other hand, property (A) implies that the probability that an exponentially distributed delay finishes instantaneously is zero ($Prob\{delay \leq 0\} = 0$). We therefore employ the *maximal progress assumption.* We assume that a process that may perform a silent action is not allowed to let time pass. The above process is therefore equal to $\tau.Q$. Since this equality is not evident from the operational rules it will become part of the definition of strong and weak bisimilarity. For this purpose, we distinguish the elements of IMC according to their ability to perform a silent action. We use $E\not\surd$ to denote *unstable* expressions satisfying $\exists F : E \xrightarrow{\tau} F$ and $E\surd$ to denote the converse. Expressions with the latter property will be called *stable* expressions in the sequel. Intuitively, only stable expressions may spend time whereas unstable expressions follow the maximal progress assumption. Note that $\surd$ can be equally defined by means of a syntactic predicate on IMC, like $\downarrow$. For expressions E that are stable as well as well-defined we use the shorthand notation $E\surd\!\downarrow$.

We are now ready to introduce strong and weak bisimilarity on IMC. As usual we define them for closed expressions, and afterwards lift them to IMC in the standard manner. The set of equivalence classes of a given equivalence relation $\mathcal{B}$ on the set IMC is denoted $\mathsf{IMC}/\mathcal{B}$. $[E]_{\mathcal{B}}$ denotes the equivalence class of $\mathcal{B}$ containing E.

Definition 5. *An equivalence relation $\mathcal{B}$ on IMP is a strong bisimulation iff $P\,\mathcal{B}\,Q$ implies*

1. *for all $a \in Act$, $P \xrightarrow{a} P'$ implies $Q \xrightarrow{a} Q'$ for some Q' with $P'\,\mathcal{B}\,Q'$,*
2. *if $P\surd\!\downarrow$ then $Q\surd\!\downarrow$ and $\gamma(P, C) = \gamma(Q, C)$ for all $C \in \mathsf{IMP}/\mathcal{B}$.*

Two processes P and Q are strongly bisimilar (written $P\sim Q$) if (P, Q) is contained in some strong bisimulation.

In this definition, maximal progress is realized because the stochastic timing behaviour (evaluated by means of γ) is irrelevant for unstable expressions. Furthermore we do not compare the timing behaviour of ill-defined processes. The reason for doing that is best explained by means of an example. An ill-defined process like $\mathrm{rec}X.(X + (\lambda).0)$ may possess an infinitely branching transition system (for each $n \in \mathbb{N}_0$ we have $\mathrm{rec}X.(X + (\lambda).0) \xrightarrow{\lambda, l^n r} 0$). Our restriction to well-defined expressions thus avoids the need to calculate and compare infinite sums of rates.

Timed versions of bisimilarity (e.g. [**?**,**?**]) usually require to cumulate subsequent time intervals. This is sometimes called *time additivity*. In our calculus, time additivity is not possible. The reason is that sequences of exponentially distributed delays are not exponentially distributed, since the class of exponential distributions is *not* closed under convolution. (There is no λ satisfying $Prob\{delay_1 + delay_2 \leq t\} = 1 - e^{\lambda t}$ if $delay_1$ and $delay_2$ are exponentially distributed.) In other words, it is impossible to replace a sequence of timed transitions by a single timed transition without affecting the probability distribution of the total delay. We thus demand that timed transitions have to be bisimulated in the strong sense, even for weak bisimilarity (in contrast to action transitions).

We let $\stackrel{a}{\Longrightarrow}$ and $\stackrel{\widehat{a}}{\Longrightarrow}$ abbreviate $\xrightarrow{\tau}^{*} \xrightarrow{a} \xrightarrow{\tau}^{*}$ except if $a = \tau$. In this case, $\stackrel{\tau}{\Longrightarrow}$ denotes $\xrightarrow{\tau}^{+}$ and $\stackrel{\widehat{\tau}}{\Longrightarrow}$ denotes $\xrightarrow{\tau}^{*}$. For a set of expressions C we define C^{τ} as the set of expressions that may silently evolve into an element of C, i.e. $C^{\tau} = \{E \mid \exists F \in C : E \stackrel{\widehat{\tau}}{\Longrightarrow} F\}$.

Definition 6. *An equivalence relation $\mathcal{B}$ on* IMP *is a weak bisimulation iff $P \,\mathcal{B}\, Q$ implies*

1. *for all $a \in Act$, $P \stackrel{\widehat{a}}{\Longrightarrow} P'$ implies $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ for some Q' with $P' \,\mathcal{B}\, Q'$,*
2. *if $P \stackrel{\widehat{\tau}}{\Longrightarrow} P' \surd\!\downarrow$ then for some $Q' \surd\!\downarrow$, $Q \stackrel{\widehat{\tau}}{\Longrightarrow} Q'$ and $\gamma(P', C^{\tau}) = \gamma(Q', C^{\tau})$ for all $C \in \mathsf{IMP}/\mathcal{B}$.*

Two processes P and Q are weakly bisimilar (written $P \approx Q$) if (P, Q) is contained in some weak bisimulation.

It can be shown that $\approx$ ($\sim$, respectively) is a weak (strong) bisimulation. We illustrate the distinguishing power of $\approx$ by means of some examples, depicted in Figure **??**. (We have used $\equiv$ to denote syntactic identity.) The first two processes, P_1 and P_2, are equivalent because P_2 is unstable (thus the μ-branch is irrelevant) but may silently evolve to a stable process that is identical (thus equivalent) to P_1. The process P_3 is equivalent to the former two, because $\gamma(P_1, ([0]_{\approx})^{\tau}) = 2\lambda = \gamma(P_3, ([0]_{\approx})^{\tau})$ and $\gamma(P_1, ([\tau.0 + a.0]_{\approx})^{\tau}) = 2\lambda = \gamma(P_3, ([\tau.0 + a.0]_{\approx})^{\tau})$ (all other values of γ are 0 in either case). In contrast, $\gamma(P_4, ([\tau.0 + a.0]_{\approx})^{\tau}) = \lambda$ whence we have that P_4 is not weakly bisimilar to the former three processes.

The shape of this last example sheds some interesting light on our definition. Assume, for the moment, that λ is just an action like all the others. Then, P_3 and P_4 would be equated under the usual notion of weak bisimilarity

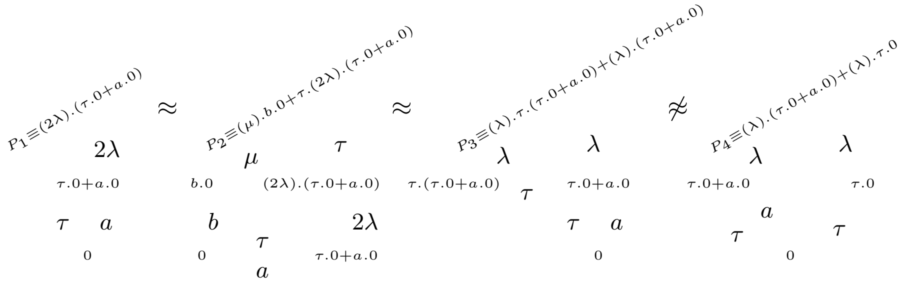

Fig. 2. Some characteristic examples for weak bisimilarity.

while they would *not* under *branching* bisimilarity. Branching bisimilarity has been introduced by van Glabbeek&Weijland [**?**]. Here, however, weak bisimilarity already distinguishes the two, because multiplicities of timed transitions are relevant. (That is the reason why $\gamma(P_1, ([0]_\approx)^\tau) = 2\lambda = \gamma(P_4, ([0]_\approx)^\tau)$ but $\gamma(P_1, ([\tau.0 + a.0]_\approx)^\tau) \neq \gamma(P_4, ([\tau.0 + a.0]_\approx)^\tau$.) In general, it is possible to reformulate weak bisimilarity such that timed transitions are treated in the same way as external transitions in branching bisimularity. This is particularly expressed in the following lemma, where the equivalence class C has replaced C^τ.

Lemma 1. *An equivalence relation $\mathcal{B}$ on IMP is a weak bisimulation iff $P\,\mathcal{B}\,Q$ implies*

1. *for all $a \in Act$, $P \xrightarrow{a} P'$ implies $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ for some Q' with $P'\,\mathcal{B}\,Q'$,*
2. *if $P\surd\downarrow$ then for some $Q'\surd\downarrow$, $Q \stackrel{\widehat{\tau}}{\Longrightarrow} Q'$ and $\gamma(P, C) = \gamma(Q', C)$ for all $C \in \mathsf{IMP}/\mathcal{B}$.*

We shall frequently use this reformulation in the sequel. Unsurprisingly, $\approx$ is not substitutive with respect to choice. We therefore proceed as usual and define the (provably) coarsest congruence contained in $\approx$.

Definition 7. *P and Q are observationally congruent, written $P \stackrel{c}{\approx} Q$, iff:*

1. *for all $a \in Act$, $P \xrightarrow{a} P'$ implies $Q \stackrel{a}{\Longrightarrow} Q'$ for some Q' with $P'{\approx}Q'$,*
2. *for all $a \in Act$, $Q \xrightarrow{a} Q'$ implies $P \stackrel{a}{\Longrightarrow} P'$ for some P' with $P'{\approx}Q'$,*
3. *$P\surd\downarrow$ (or $Q\surd\downarrow$) implies $\gamma(P, C) = \gamma(Q, C)$ for all $C \in \mathsf{IMP}/{\approx}$*
4. *$P\surd\downarrow$ iff $Q\surd\downarrow$.*

Definition 8. *Let $\mathcal{R} \subseteq \mathsf{IMP} \times \mathsf{IMP}$. We extend $\mathcal{R}$ to $\mathsf{IMC} \times \mathsf{IMC}$ as follows. Let E, $F \in \mathsf{IMC}$. Then $E\,\mathcal{R}\,F$ iff $\forall P_1, \ldots, P_n \in \mathsf{IMP} : E\{\boldsymbol{P}/\boldsymbol{X}\}\,\mathcal{R}\,F\{\boldsymbol{P}/\boldsymbol{X}\}$, where $\boldsymbol{X}$ denotes the vector (of length n) of variables occurring free in E or F, and $\{\boldsymbol{E}/\boldsymbol{X}\}$ denotes the simultaneous substitution of each X_i by E_i.*

It can be shown that $\approx\ \supset \stackrel{c}{\approx} \supset \sim$. In addition, strong bisimilarity and observational congruence are compositional relations indeed. The following theorem follows from Theorem **??** in the next section.

Theorem 1. *$\stackrel{c}{\approx}$ is a congruence with respect to the operators of IMC.*

3 Axiomatisation

In this section we develop a set of equational laws that is sound and complete with respect to $\stackrel{c}{\approx}$. To achieve completeness is by far not straightforward, due to the presence of maximal progress. Divergent expressions, performing an infinite number of silent steps (e.g. $\text{rec}X.(\tau.X + (\lambda).0))$, will be our main concern. In ordinary CCS the KFAR law $\text{rec}X.(\tau.X + E) = \text{rec}X.\tau.E$ is responsible to remove such infinite sequences. This law is not sound in our calculus. To illustrate this phenomenon suppose $\text{rec}X.(\tau.X + (\lambda).0) \stackrel{c}{\approx} \text{rec}X.(\tau.(\lambda).0)$. This implies $\text{rec}X.(\tau.X + (\lambda).0) \approx (\lambda).0$. But, since $(\lambda).0\surd\downarrow$ there must be some $P\surd\downarrow$ with $\text{rec}X.(\tau.X + (\lambda).0) \stackrel{\hat{\tau}}{\Longrightarrow} P$ which is not the case. Hence, we are forced to treat such loops of silent actions in a different way. We make them explicit by means of a distinguished symbol $\perp$ indicating ill-defined expressions. We equate divergent and ill-defined expressions. This is inspired by [?], but divergence (and ill-definedness) can be abstracted away if a silent computation is possible, i.e., $\perp + \tau.E = \tau.E$. The symbol $\perp$ is not part of the language IMC we are aiming to axiomatise. It will however be an essential part of the laws. For instance, in order to equate the expressions $\text{rec}X.(\tau.X + \tau.0)$ and $\tau.0$ the symbol $\perp$ appears (and vanishes again) inside the proof. We therefore define an extended language $\mathsf{IMC}^\perp$ as follows:

Definition 9. *Let $\lambda \in \mathbb{R}$, $a \in Act$ and $X \in Var$. We define the language $\mathsf{IMC}^\perp$ as the set of expressions given by the following grammar.*

$$\mathcal{E} ::= 0 \quad | \quad \perp \quad | \quad (\lambda).\mathcal{E} \quad | \quad a.\mathcal{E} \quad | \quad \mathcal{E} + \mathcal{E} \quad | \quad X \quad | \quad \text{rec}X.\mathcal{E}.$$

All definitions introduced in Section 2 can be equally defined for this langage. Note that no transitions are derivable for $\perp$ by means of the operational rules in Figure ??. $\mathsf{IMC}^\perp_\downarrow \subset \mathsf{IMC}^\perp$ denotes the set that is obtained from Definition ??, applied to $\mathsf{IMC}^\perp$. Note that $\perp \notin \mathsf{IMC}^\perp_\downarrow$. Similarly, Definition ?? yields the notion of a weak bisimulation on $\mathsf{IMP}^\perp$. Then $\stackrel{\perp}{\approx}$ is the union of all weak bisimulations on $\mathsf{IMP}^\perp$. Finally, Definition ?? (with $\approx$ replaced by $\stackrel{\perp}{\approx}$) yields an observational congruence $\stackrel{c\perp}{\approx} \subset \mathsf{IMP}^\perp \times \mathsf{IMP}^\perp$ with $\stackrel{c\perp}{\approx} \subset \stackrel{\perp}{\approx}$. By replacing in Definition ?? IMP and IMC by $\mathsf{IMP}^\perp$ and $\mathsf{IMC}^\perp$, respectively, $\stackrel{\perp}{\approx}$ and $\stackrel{c\perp}{\approx}$ can be lifted to $\mathsf{IMC}^\perp$.

Theorem 2. *$\stackrel{c\perp}{\approx}$ is a congruence with respect to the operators of $\mathsf{IMC}^\perp$. Furthermore, for $E, F \in \mathsf{IMC}$, $E \approx F$ iff $E \stackrel{\perp}{\approx} F$, and $E \stackrel{c}{\approx} F$ iff $E \stackrel{c\perp}{\approx} F$.*

Because of the first statement above it is justified to call $\stackrel{c\perp}{\approx}$ an observational congruence. Because of the second statement every proof system for $\stackrel{c\perp}{\approx}$ can equally be used as a proof system for $\stackrel{c}{\approx}$. Finally both statements together prove Theorem ??.

We are now ready to introduce a proof system for $\stackrel{c\perp}{\approx}$ on $\mathsf{IMC}^\perp$ (and thus for $\stackrel{c}{\approx}$ on IMC). Figure ?? lists relevant axioms grouped into different sets. We omit the usual rules for structural congruence. The axioms of $\mathcal{A} \cup \mathcal{A}^{\text{rec}}$ together with the idempotence law

$$(I)\ E + E = E$$

Axiom system $\mathcal{A}$

$(B1)$	$E + 0 = E$	$(\tau 1)$	$a.\tau.E = a.E$
$(B2)$	$E + F = F + E$	$(\tau 2)$	$E + \tau.E = \tau.E$
$(B3)$	$(E + F) + G = E + (F + G)$	$(\tau 3)$	$a.(E + \tau.F) + a.F = a.(E + \tau.F)$

Axiom system $\mathcal{A}^{\text{rec}}$

(rec1) $\text{rec}X.E = \text{rec}Y.(E\{Y/X\})$
provided that Y is not free in $\text{rec}X.E$.
(rec2) $\text{rec}X.E = E\{\text{rec}X.E/X\}$
(rec3) $F = E\{F/X\}$ implies $F = \text{rec}X.E$
provided that X is strongly guarded in E.

Axiom system $\mathcal{A}^{I^*}$

$(I1)$ $a.E + a.E = a.E$	$(I2)$ $E + E + \perp + \perp = E + \perp$

Axiom system $\mathcal{A}^{\lambda}$

$(I3)$ $(\lambda).E + (\mu).E = (\lambda + \mu).E$	$(\tau 4)$ $(\lambda).\tau.E = (\lambda).E$

Axiom system $\mathcal{A}^{\perp}$

$(\perp 1)$ $(\lambda).E + \perp = \perp$	$(\perp 2)$ $\perp + \tau.E = \tau.E$

(rec4)	$\text{rec}X.(X + E) = \text{rec}X.(\perp + E)$
(rec5)	$\text{rec}X.(\tau.X + E) = \text{rec}X.(\tau.(\perp + E))$
(rec6)	$\text{rec}X.(\tau.(X + E) + F) = \text{rec}X.(\tau.X + E + F)$

Fig. 3. Axioms for observational congruence.

are standard laws forming a complete proof system of observational congruence for strongly guarded regular CCS [**?**]. Our axiomatisation is based on this system, but with a slight modification. We require to replace idempotence by a set of laws $\mathcal{A}^{I^*}$. This refinement[2] is needed because of the presence of stochastic time [**?**]. Delay rate quantities have to be cumulated according to property (C) in order to represent the stochastic timing behaviour of expressions like $(\lambda).E + (\lambda).E$.

Soundness As we will see, the axiom system $\widehat{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}^{\text{rec}} \cup \mathcal{A}^{I^*} \cup \mathcal{A}^{\lambda} \cup \mathcal{A}^{\perp}$ is sound and complete for $\mathsf{IMC}^{\perp}$ modulo observational congruence. The system $\mathcal{A}^{\lambda}$ is a collection of laws that cover the impact of stochastic time in $\mathsf{IMC}^{\perp}$. Law $(I3)$ axiomatises property (C) (and is the reason why (I) is invalidated in general) while law $(\tau 4)$ is an obvious adaption of $(\tau 1)$. Note that an adaption of law $(\tau 3)$, the distinguishing law between weak and branching bisimilarity [**?**], is not required, as a consequence of Lemma **??**.

The most interesting aspect of our proof system is the treatment of divergence and ill-definedness reflected in $\mathcal{A}^{\perp}$. Law $(\perp 1)$ expresses that ill-definedness makes

[2] Note, however, that $\mathcal{A} \cup \mathcal{A}^{\text{rec}} \cup \{(I1)\}$ gives rise to a complete proof system of observational congruence for strongly guarded regular CCS.

irrelevant the passage of time. Law ($\perp 2$) is the key to escape ill-definedness by means of a silent alternative.

Law (rec4) states that fully unguardedness is ill-defined. The last two laws for recursion explicitly handle divergent expressions that may perform an infinite number of silent steps. Law (rec4) and (rec6) are taken from [**?**], while law (rec5) is *not*. It replaces Milner's KFAR axiom by basically equating divergence and ill-definedness for loops of length 1. Law (rec6) reduces the length of loops of silent steps such that they can eventually be handled by (rec5). The laws (rec5) and (rec6) are essential in order to handle weakly guarded expressions that are not strongly guarded. We shall write $\widehat{\mathcal{A}} \vdash E = F$ if $E = F$ may be proved from $\widehat{\mathcal{A}}$.

It is worth to point out that WFAR, the fair abstraction rule of unstable divergence à la Bergstra *et al.* [**?**], is valid in the presence of maximal progress while KFAR is not sound. In our setting, a WFAR axiom can be formulated as follows:

$$\mathrm{rec}X.(\tau.X + \tau.E + F) = \mathrm{rec}X.(\tau.(\tau.E + F)),$$

and can be derived by means of law (rec5) and ($\perp 2$), i.e.

$$\widehat{\mathcal{A}} \vdash \mathrm{rec}X.(\tau.X + \tau.E + F) = \mathrm{rec}X.(\tau.(\perp + \tau.E + F)) = \mathrm{rec}X.(\tau.(\tau.E + F)).$$

This derivation is indeed a simple example where the symbol $\perp$ appears and vanishes inside a proof. Another notable example is the maximal progress axiom mentioned in the introduction, requiring law ($\perp 1$) and ($\perp 2$).

Lemma 2. *For $E, F \in \mathsf{IMC}^{\perp}$ it holds that* $\widehat{\mathcal{A}} \vdash (\lambda).E + \tau.F = \tau.F$.

Turning our attention to the adequacy of this proof system to decide $\overset{c}{\approx}^{\perp}$, we first state that $\widehat{\mathcal{A}}$ is indeed sound with respect to observational congruence on $\mathsf{IMC}^{\perp}$.

Theorem 3. *For $E, F \in \mathsf{IMC}^{\perp}$ it holds that* $\widehat{\mathcal{A}} \vdash E = F$ *implies* $E \overset{c}{\approx}^{\perp} F$.

Completeness In order to address the question whether our set of laws is complete, i.e. sufficiently powerful to allow the deduction of all semantic equalities, we closely follow the lines of Milner [**?**] and use standard equation sets (SES), i.e. mutually recursive systems of defining equations, to capture the impact of rec for strongly guarded expressions. Nontrivial transformations are needed to prove the following theorem. In particular, it has to be assured that two separate SES, each satisfied by some strongly guarded expression, can be merged into a single SES if both expressions are observational congruent. Verifying this is a lot more involved than the usual proofs owed to the presence of stochastic timing and maximal progress. The details can be found in [**?**].

Theorem 4. *For strongly guarded $E, F \in \mathsf{IMC}^{\perp}$, $E \overset{c}{\approx}^{\perp} F$ implies $\widehat{\mathcal{A}} \vdash E = F$.*

Let us now extend this result beyond strongly guarded expressions, i.e. expressions where every recursive variable is preceded by an action prefix different from τ or a delay prefix (λ). In CCS weakly guarded expressions are easily handled, because KFAR can be used to remove loops of τs. As discussed above the presence of maximal progress does not allow this treatment since loops of τs cause divergence, except if a silent alternative exists. On a syntactic level this property is reflected by the laws (rec4)-(rec6). These laws are indeed sufficient to deduce all semantic equalities that involve unguardedness. Once again, space constraints urge us to refer to [?] for the details of the proof. The structure resembles the proof in [?], but the arguments subtly differ.

Theorem 5. *For each $E \in \mathsf{IMC}^{\perp}$ there exists a strongly guarded $F \in \mathsf{IMC}^{\perp}$ such that $\widehat{\mathcal{A}} \vdash E = F$.*

Theorem **??** and Theorem **??** provide the necessary means to derive completeness for arbitrary expressions of $\mathsf{IMC}^{\perp}$.

Theorem 6. *Let $E, F \in \mathsf{IMC}^{\perp}$. $E \overset{c}{\approx}^{\perp} F$ implies $\widehat{\mathcal{A}} \vdash E = F$.*

Corollary 1. *For $E, F \in \mathsf{IMC}^{\perp}$, $\quad E \overset{c}{\approx}^{\perp} F$ iff $\widehat{\mathcal{A}} \vdash E = F$.*

4 Discussion and Applications

Observational congruence treats divergence in the style of WFAR, it allows to escape divergence only if a silent alternative exists. It is interesting to discuss $\overset{c}{\approx}$ in the context of ordinary CCS that arises from IMC by disallowing delay prefixing. We use $\mathsf{IMC}_{\not\chi}$ to denote this subset of IMC. With the technical means of the previous section the following result is easy to show.

Theorem 7. *For $E, F \in \mathsf{IMC}_{\not\chi}$, $(\mathcal{A} \cup \{(I)\} \cup \mathcal{A}^{\text{rec}} \cup \mathcal{A}^{\perp} \setminus \{(\perp 1)\}) \vdash E = F$ iff $E \overset{c}{\approx} F$.*

Stated differently, we have obtained a complete proof system for CCS modulo observational congruence with WFAR. The proof system differs from other treatments of divergence in CCS. Walker has studied divergence sensitive bisimilarity [?] (see also [?]). His basic notion is a preorder rather than an equivalence. The induced equivalence turns out to be incomparable with Milner's original notion of observational congruence.

Our notion of observational congruence does neither coincide with Milner's divergence insensitive notion (denoted $\overset{c}{\approx}_{\text{Milner}}$) nor with Walker's divergence sensitive variant ($\overset{c}{\approx}_{\text{Walker}}$). Roughly, the reason is that, different from Walker, it *is* possible to escape from *unstable* divergence. But, deviating from Milner, it *is not* possible to escape from *stable* divergence. As a whole, it can be shown that $\overset{c}{\approx}$ is incomparable with Walker's notion (cf. the first and last pair in Figure **??** [?]). In contrast, $\overset{c}{\approx}$ turns out to be finer than Milner's observational congruence.

Theorem 8. *For $E, F \in \mathsf{IMC}_{\not\chi}$ it holds that $\quad E \overset{c}{\approx} F \quad$ implies $\quad E \overset{c}{\approx}_{Milner} F$.*

The inclusion is strict, as testified by the middle pair in Figure **??**.

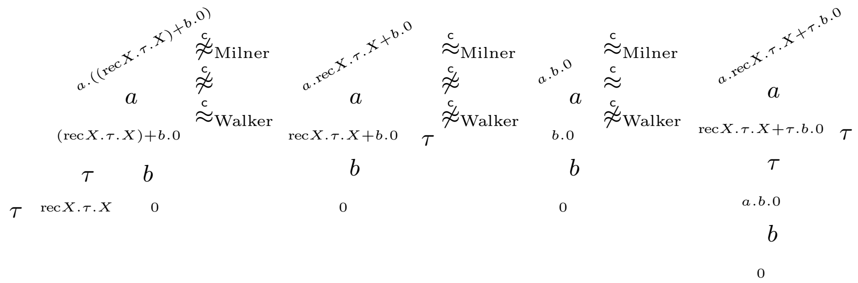

Fig. 4. Observational Congruence is finer than Milner's notion and incomparable with Walker's.

Applications The proof system given in this paper can be adapted to establish formerly unknown sound and complete proof systems for a variety of process calculi with (stochastic) time or with priority. For this purpose the notion of well-definedness is incorporated into the respective definition of bisimilarity. We briefly sketch some results.

- The set of laws $\{(B1),(B2),(B3),(I1),(I2),(I3),(\bot 1),(\bot 2),(\text{rec4})\} \cup \mathcal{A}^{\text{rec}}_{\sim}$ provides a sound and complete proof system of strong bisimilarity (Definition **??**). $\mathcal{A}^{\text{rec}}_{\sim}$ is obtained from $\mathcal{A}^{\text{rec}}$ by changing the word 'strongly' to 'weakly' in law (rec3).
- The set of laws $\{(B1),(B2),(B3),(I2),(I3),(\bot 1),(\text{rec4})\} \cup \mathcal{A}^{\text{rec}}_{\sim}$ provides a sound and complete proof system of strong equivalence of PEPA [**?**] (where '$(a, \_);$' replaces '$(\_).$' in law $(I3)$ and law $(\bot 1)$). The same set of laws is sound and complete for Markovian bisimilarity of MTIPP [**?**] (giving an implicit proof that strong equivalence and Markovian bisimilarity agree on this common fragment).
- The set of laws $\{(B1),(B2),(B3),(I),(\bot 1),(\bot 2),(\text{rec4})\} \cup \mathcal{A}^{\text{rec}}_{\sim}$ (where the low priority prefix '$\underline{a}.$' replaces '$(\lambda).$' in each of the laws) is sound and complete with respect to strong congruence on CCS^{prio}, the prioritized calculus of [**?**]. In particular, Lemma **??** becomes the priority axiom mentioned in our introduction.
- For a simplified variant of prioritised observational conguence on CCS^{prio}, $\mathcal{A} \cup \{(I)\} \cup \mathcal{A}^{\text{rec}} \cup \mathcal{A}^{\bot} \cup \{(\tau 4)\} \cup \{(\lambda).(E+\tau.F)+(\lambda).F = (\lambda).(E+\tau.F)\}$ is a sound and complete proof system (where again each '$(\lambda).$' has to be replaced by '$\underline{a}.$'). This prioritised observational congruence is weak in the sense that it abstracts from sequences of silent *high* priority actions. *Low* prioritised silent actions, however, are treated as in strong bisimulation. This is the main simplification with respect to the approach of [**?**] where most of the complexity is due to a weak transition relation that involves silent actions of both, high and low priority.

The details are carried out in [**?**], respectively [**?**]. Further completeness results appear feasible. Of particular interest is the timed calculus CSA [**?**], especially if restricted to a single clock. This fragment of CSA agrees with Hennessy and Regan's TPL [?], but originally TPL is developed in a testing setting. We conjecture, that a sound and complete proof system for observational congruence for this fragment can be based on $\mathcal{A}^{\text{rec}} \cup \mathcal{A}^{\perp}$. Another obvious candidate for an adaption of our proof system is EMPA [?], since it is strongly inspired by both PEPA and MTIPP.

Parallel composition For simplicity, the language IMC does not possess means to express parallel composition of expressions. We will outline that parallel composition can be easily added to IMC, due to the separation of actions and delays. Different from our calculus, MTIPP, PEPA, and EMPA *replace* action-prefix $a.E$ by $(a, \lambda).E$. The basic difference between these algebras is the calculation of the resulting rate in case of synchronisation. MTIPP proposes the product of rates, EMPA forbids this type of synchronisation and requires one agent to determine the rate only while the other components need to be passive (i.e. willing to accept any rate), and finally PEPA computes the maximum of mean delays while incorporating the individual synchronisation capacities of agents.

None of these algebras uses the maximum of delays, since the class of exponential distributions is not closed under maximum. However, the absence of this closure property does not pose a problem for IMC since — due to the separation of the Markovian and action transitions — we can model the maximum of two exponential distributions explicitly (as a corresponding phase-type distribution).

For instance, CCS parallel composition '|' (as well as relabelling and restriction) can be easily added to the calculus as well as the proof system, as in [?]. The only particularity that has to be clarified is the semantics of delayed expressions. Indeed, property (B) justifies to simply interleave delays, i.e. extending the definition of $\dashrightarrow$ (Definition ??) essentially by

$$\frac{E \overset{\lambda,\, w}{\dashrightarrow} E'}{E|F \overset{\lambda,\, l\,w}{\dashrightarrow} E'|F} \qquad\qquad \frac{F \overset{\lambda,\, w}{\dashrightarrow} F'}{E|F \overset{\lambda,\, r\,w}{\dashrightarrow} E|F'}$$

(plus the standard rules for action transitions). In the same way we can establish an expansion law that allows to equate

$$(\lambda).E \mid (\mu).F \quad = \quad (\lambda).(\ E \mid (\mu).F\ ) + (\mu).(\ (\lambda).E \mid F\ ).$$

So, the complete proof system introduced in Section 3 can be straightforwardly extended to cover the usual operators of a CCS based process algebra, and also to CSP or LOTOS style operators [?]. Note that the separation of delays and actions simplifies the semantics of parallel composition, but is not a prerequisite to achieve an expansion law, since for instance, an expansion law for MTIPP style synchronisation is known [?]. Nevertheless this separation appears to be necessary to obtain the maximum of delays in the synchronisation case. We consider this solution as a natural choice, like many others do, e.g. [?,?].

5 Concluding remarks

In this paper we have investigated weak bisimilarity and observational congruence in a stochastic timed calculus with maximal progress. Our notions refine the usual notions on CCS because they allow to escape from divergence (only) if a silent alternative exists. This takes the effect of WFAR. The refinement is needed in order to capture the interplay of maximal progress and divergence. We have obtained a sound and complete proof system for arbitrary (including unguarded) expressions. Since Milner's law $\mathrm{rec}X.\tau.X + P = \mathrm{rec}X.\tau.P$ is invalidated by maximal progress we have replaced it by a set of laws that allow abstraction of unstable divergence.

The algebra IMC contains both (homogeneous) continuous time Markov chains and CCS as proper subalgebras. Since our treatment of divergence is orthogonal to the aspects of stochastic time, it can be profitably adapted to other calculi with maximal progress or with a notion of priority. We have highlighted how some of the existing gaps of incomplete proof systems are filled by adapting the techniques of this paper. As far as we know, this paper is first to solve the open problem of complete proof systems for observational congruence for calculi including either priority or maximal progress.

As part of the TIPP project, IMC has been extended in the direction of LOTOS, and this extension has been applied to study performance properties of parallel and distributed systems, see [**?**]. For the purpose of compositional analysis we have recently adapted well-known partition refinement algorithms [**?**] for computing strong and weak bisimilarity on IMC [**?**]. The computational complexity of these relations is not increased when moving from CCS to the setting of IMC.

It is well known that many strong and weak equivalences can be characterised by means of simple modal logic characterisations. We plan to investigate such characterisations for the equivalence notion discussed here. This would be beneficial for the specification and verification of particular stochastic timed properties. Currently, properties of an IMC specification are evaluated by transforming the transition system into a Markov chain and subsequent calculation of state probabilities. The interpretation of these probabilities is not easy because the behavioural view is lost on the level of the Markov chain. Even though of a speculative nature, we would prefer a model checking approach to this problem, inspired by [**?**].

References

1. L. Aceto and A. Jeffrey. A Complete Axiomatization of Timed Bisimulation for a Class of Timed Regular Behaviours. *Theoretical Computer Science* 152, 1995.
2. H. R. Andersen and M. Mendler. An Asynchronous Process Algebra with Multiple Clocks. In Proc. *ESOP 94*, Springer LNCS 788:58-73, 1994.
3. A. Aziz, K. Sanwal, V. Singhal, and R.K. Brayton. *Verifying Continuous Time Markov Chains*. In Proc. *CAV 96*, Springer LNCS 1102:269-276,1996.
4. J.C.M. Baeten and J.A. Bergstra. Real Time Process Algebra. *Formal Aspects of Computing* 3:142-188, 19991.
5. J.C.M. Baeten, J.A. Bergstra, and J.W. Klop. On the Consistency of Koomen's Fair Abstraction Rule. *Theoretical Computer Science* 51:129-176, 1987.
6. J.A. Bergstra, J.W. Klop, and E.-R. Olderog. Failures Without Chaos: A New Process Semantics for Fair Abstraction. In *Formal Description of Programming Concepts - III.* Elsevier, 1987.
7. M. Bernardo and R. Gorrieri. Extended Markovian Process Algebra. In Proc. *CONCUR 96*, Springer LNCS 1119:315-330, 1996.

8. E. Brinksma, A. Rensink, and W. Vogler. Fair Testing. In Proc. *CONCUR 95*, Springer LNCS 962:313-327, 1995.
9. R. Cleaveland and M. Hennessy. Priorities in Process Algebra. *Information and Computation* 87:58-77,1990.
10. R. Cleaveland, G. Lüttgen, and M. Mendler. An Algebraic Theory of Multiple Clocks. In Proc. *CONCUR 97*, Springer LNCS 1243:166-180, 1997.
11. R. Cleaveland, G. Lüttgen, and V. Natarjan. A Process Algebra with Distributed Priorities. In Proc. *CONCUR 96*, Springer LNCS 1119:34-49,1996.
12. P.R. D'Argenio, J.-P. Katoen and E. Brinksma. An algebraic approach to the specification of stochastic systems (extended abstract). *Proc. Working Conference on Programming Concepts and Methods*. Chapman & Hall, 1998.
13. W. Feller.*An Introduction to Probability Theory and Its Applications*. John Wiley & Sons, 1968.
14. W. Fokkink. An Axiomatization for Regular Processes in Timed Branching Bisimulation. Fundamenta Informaticae 32(3/4): 329-340, 1998.
15. R. van Glabbeek and W. Weijland. Branching Time and Abstraction in Bisimulation Semantics. *Journal of the ACM* 43(3):555-600,1996.
16. R.J. van Glabbeek, S.A. Smolka, and B. Steffen. Reactive, Generative, and Stratified Models of Probabilistic Processes. *Information and Computation* 121, 1995.
17. N. Götz, U. Herzog, and M. Rettelbach. Multiprocessor and distributed system design: The integration of functional specification and performance analysis using stochastic process algebras. In *PERFORMANCE 93*, Springer LNCS 729, 1993.
18. M. Hennessy and G.D. Plotkin. *A term model for CCS*. In Proc. *MFCS 80*, Springer LNCS 88:261-274, 1980.
19. M. Hennessy and T. Regan. A process algebra for timed systems. *Information and Computation*, 117(2):221-239, 1995.
20. H. Hermanns. Interactive Markov Chains. PhD thesis, Universität Erlangen-Nürnberg, to appear 1998.
21. H. Hermanns, and M. Lohrey. Observational Congruence in a Stochastic Timed Calculus with Maximal Progress. Tech. Rep. IMMD-VII/7-97, Universität Erlangen, 1997.
22. H. Hermanns, M. Rettelbach, and T. Weiß. Formal Characterisation of Immediate Actions in an SPA with Non-Deterministic Branching. *The Computer Journal*, 38(7):530-541, 1995.
23. H. Hermanns, U. Herzog, and V. Mertsiotakis. Stochastic Process Algebras – Between LOTOS and Markov Chains. *Computer Networks and ISDN Systems*, 30 (9/10):901–924, 1998.
24. H. Hermanns and M. Rettelbach. Syntax, Semantics, Equivalences, and Axioms for MTIPP. In *Proc. of the 2nd Workshop on Process Algebras and Performance Modelling*, Erlangen-Regensberg, July 1994. IMMD, Universität Erlangen.
25. J. Hillston. *A Compositional Approach to Performance Modelling*. Cambridge University Press. 1996.
26. K.G. Larsen and A. Skou. Bisimulation through Probabilistic Testing. *Information and Computation* 94:1-28, 1991.
27. C.J. Koomen. Algebraic Specification and Verification of Communication Protocols. *Science of Computer Programming* 5:1-36, 1985.
28. R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.
29. R. Milner. A Complete Axiomatization for Observational Congruence of Finite-State Behaviours. *Information and Computation* 81:227-247, 1989.
30. F. Moller and C. Tofts. A Temporal Calculus of Communicating Systems. In Proc. *CONCUR 90*, Springer LNCS 458:401-415, 1990.
31. V. Natarjan and R. Cleaveland. Divergence and Fair Testing. In Proc. *ICALP 95*, Springer LNCS 944:648-659, 1995.
32. V. Natarajan, I.Christoff, L.Christoff, and R.Cleaveland. Priorities and Abstraction in Process Algebra. In Proc. *FST&TCS 94*, Springer LNCS 880:217-230, 1994.
33. X. Nicollin and J. Sifakis. An Overview and Synthesis on Timed Process Algebras. In *Real-Time: Theory in Practice*, Springer LNCS 600, 1991.
34. X. Nicollin and J. Sifakis. The Algebra of Timed Processes ATP: Theory and Application. *Information and Computation* 114:131-178, 1991.
35. W.J. Stewart. *Introduction to the Numerical Solution of Markov Chains*. Princeton University Press, 1994.

36. R. Paige and R. Tarjan. Three Partition Refinement Algorithms. *SIAM Journal of Computing*, 16(6):973–989, 1987.
37. D.J. Walker. Bisimulation and Divergence. *Information and Computation* 85:202-241, 1990.
38. W. Yi. CCS + Time = An Interleaving Model for Real Time Systems. In Proc. *ICALP 91*, Springer LNCS 510:217-228, 1991.